A SIMPLE MODEL OF THE FAILURE OF THE EXPECTATIONS HYPOTHESIS

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Abstract. This paper presents a simple exponential-affine model of the term structure that replicates the failure of the expectations hypothesis quite well. The model has two Gaussian state variables. Under the equivalent martingale measure, both state variables (the short rate and its stochastic mean) determine the shape of the yield curve, while under the physical measure the short rate is Markovian. Thus instantaneous forward rates have a classical errors-in-variables structure: They are moved by two variables (that are independent under the physical measure), but the short-term interest rate is not moved by one of them. This model shows that stochastic volatility is not required to model the failure of the expectations hypothesis. The key is stochastic risk premia.

1. Introduction

In its strong form, the expectations hypothesis asserts that instantaneous forward rates are the conditional expectations of the future short term interest rate. Fisher and Gilles (1998) have shown that when the strong form of the expectations hypothesis holds and there are a finite number of Markovian state variables, the yield curve is bizarre: It is a sine wave with random amplitude and phase. A weaker form of the expectations hypothesis does not require forward rates to be unbiased predictors of future spot rates. It requires only that the bias of forward rates be a deterministic function of maturity. This weak form of the expectations hypothesis can be modeled with state-independent volatilities and risk premia.[1] Under the weak form, regressions of the change in bond yields on the slope of the yield curve will produce a slope coefficient of unity. For U.S. data, this implication of the weak form has been consistently rejected. See for example Campbell and Shiller (1991).

This paper is a complementary modeling exercise to Fisher and Gilles (1998). Here I present a simple model that fits well the empirical failure of the weak from

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[1]See Campbell (1986).

of the expectations hypothesis. I show that the key to modeling the failure of the weak form is to make the price of interest-rate risk depend on a mean-reverting state variable that is independent of the interest rate.

We can embed this pure-finance model in a general-equilibrium framework and provide an interpretation in terms of the behavior of the monetary authority. In the simplest setting utility depends additively on the log of real balances, and consequently the nominal price of risk equals the volatility of the nominal money supply plus the volatility of the short-term nominal interest rate. In this general-equilibrium framework, the monetary authority sets the short-term interest rate as a function of a set of state variables. This interest-rate rule, however, does not completely determine the dynamics of nominal money growth. If additional state variables are required to describe the volatility of money growth, then the nominal price of risk will have a component that is independent of the interest rate.

2. The model

The general setting. The case I will consider falls in the following more general setting. There is a vector of d Markovian state variables, $X(t)$, that evolve according to the following stochastic differential equations (SDEs):

$$dX(t) = \mu_X(X(t))\,dt + \sigma_X(X(t))\,dW(t), \tag{2.1}$$

where W is a vector of d independent Brownian motions. The short-term risk-free interest rate and the price of risk vector are determined by functions of the state variables:

$$r(t) = \mathcal{R}(X(t)) \quad \text{and} \quad \lambda(t) = \mathcal{L}(X(t)). \tag{2.2}$$

Under the standard equivalent martingale measure the drift of the state variables is given by $\widehat{\mu}_X(X(t))$, where

$$\widehat{\mu}_X(x) = \mu_X(x) - \sigma_X(x)\,\mathcal{L}(x). \tag{2.3}$$

Duffie and Kan (1996) show that as long as $\mathcal{R}(x)$, $\widehat{\mu}_X(x)$, and $\sigma_X(x)\,\sigma_X(x)^\top$ are affine in x, then (subject to the existence of a solution to the SDEs under the equivalent martingale measure) the price at time t of a zero-coupon bond that pays one unit at time T is given by $p(t,T) = P(X(t), T-t)$, where

$$P(x,\tau) = \exp\left(-A(\tau) - B(\tau)^\top x\right), \tag{2.4}$$

where $B(\tau)$ is a vector of factor loadings. Note that since $p(T,T) = P(X(T),0) = 1$, we must have $A(0) = B_i(0) = 0$ for all i.

In particular, if

$$\mathcal{R}(x) = \mathcal{R}_0 + \mathcal{R}_1^\top x, \quad \hat{\mu}_X(x) = \hat{a}_X + \hat{b}_X\,x, \quad \text{and} \quad \sigma_X(x)\,\sigma_X(x)^\top = G_0 + \sum_{i=1}^{d} G_i\,x, \tag{2.5}$$

then

$$A'(\tau) = \mathcal{R}_0 + \hat{a}_X\,B(\tau) - \frac{1}{2}\,B(\tau)^\top G_0\,B(\tau) \tag{2.6}$$

and

$$B'(\tau) = \mathcal{R}_1 + \hat{b}_X\, B(\tau) - \frac{1}{2} \begin{pmatrix} B(\tau)^\top G_1\, B(\tau) \\ \vdots \\ B(\tau)^\top G_d\, B(\tau) \end{pmatrix}. \tag{2.7}$$

If $G_i = 0$ for $i = 1, \ldots, d$, then the distribution of state variables is Gaussian.

The particular model. There are two state variables,

$$X(t) = \begin{pmatrix} r(t) \\ z(t) \end{pmatrix}.$$

The short-term interest rate is r: $r(t) = \mathcal{R}(r(t), z(t))$, where $\mathcal{R}(r, z) = r$. Under the physical measure the two state variables are independent. Let

$$\mu_X(r, z) = \begin{pmatrix} \kappa_r\, \theta_r \\ \kappa_z\, \theta_z \end{pmatrix} + \begin{pmatrix} -\kappa_r & 0 \\ 0 & -\kappa_z \end{pmatrix} \begin{pmatrix} r \\ z \end{pmatrix} \quad \text{and} \quad \sigma_X(r, z) = \sigma_X = \begin{pmatrix} \sigma_r & 0 \\ 0 & \sigma_z \end{pmatrix},$$

where κ_i, σ_i, and θ_i are constant parameters. I assume $\kappa_i > 0$ and $\theta_i > 0$. The stochastic differential equations (SDEs) for the state variables are given by

$$dr(t) = \kappa_r\, (\theta_r - r(t))\, dt + \sigma_r\, dW_1(t) \tag{2.8a}$$

$$dz(t) = \kappa_z\, (\theta_z - z(t))\, dt + \sigma_z\, dW_2(t), \tag{2.8b}$$

where W_1 and W_2 are independent Brownian motions. The state variables are independently conditionally normally distributed. The conditional expectations and variances are given by

$$E_t\left[\begin{pmatrix} r(t+\tau) \\ z(t+\tau) \end{pmatrix}\right] = \begin{pmatrix} (1 - e^{-\kappa_r\, \tau})\, \theta_r \\ (1 - e^{-\kappa_z\, \tau})\, \theta_z \end{pmatrix} + \begin{pmatrix} e^{-\kappa_r\, \tau} & 0 \\ 0 & e^{-\kappa_z\, \tau} \end{pmatrix} \begin{pmatrix} r(t) \\ z(t) \end{pmatrix} \tag{2.9}$$

and

$$\mathrm{Var}_t\left[\begin{pmatrix} r(t+\tau) \\ z(t+\tau) \end{pmatrix}\right] = \begin{pmatrix} \frac{\sigma_r^2\, (1 - e^{-2\,\kappa_r\, \tau})}{2\, \kappa_r} & 0 \\ 0 & \frac{\sigma_z^2\, (1 - e^{-2\,\kappa_z\, \tau})}{2\, \kappa_z} \end{pmatrix}. \tag{2.10}$$

The unconditional means and variances are given by

$$\theta = \begin{pmatrix} \theta_r \\ \theta_z \end{pmatrix} \quad \text{and} \quad \mathcal{V} = \begin{pmatrix} \frac{\sigma_r^2}{2\, \kappa_r} & 0 \\ 0 & \frac{\sigma_z^2}{2\, \kappa_z} \end{pmatrix}$$

The key feature of the model is the price of risk vector, which I model using an idea from Chacko (1997). Let the price of risk function be given by

$$\mathcal{L}(r, z) = \begin{pmatrix} \dfrac{\kappa_r\, (\theta_r - r) - \widehat{\kappa}_r\, (z - r)}{\sigma_r} \\ \dfrac{\kappa_z\, (\theta_z - z) - \widehat{\kappa}_z\, (\widehat{\theta}_z - z)}{\sigma_z} \end{pmatrix}, \tag{2.11}$$

where $\widehat{\kappa}_i$ and $\widehat{\theta}_z$ are constant parameters. Ordinarily, in a Gaussian model, the price of risk is not state-dependent. Without such state-dependence, the weak form of the expectations hypothesis will hold. Also note that the price of interest-rate risk is driven (in part) by the second factor. (The model does a good job even if the price of second-factor risk is constrained to be zero.)

Following (2.3), the risk-adjusted drift is given by

$$\widehat{\mu}_X(r,z) = \begin{pmatrix} 0 \\ \widehat{\kappa}_z\,\widehat{\theta}_z \end{pmatrix} + \begin{pmatrix} -\widehat{\kappa}_r & \widehat{\kappa}_z \\ 0 & -\widehat{\kappa}_z \end{pmatrix} \begin{pmatrix} r \\ z \end{pmatrix}.$$

The SDEs under the equivalent martingale measure are given by

$$dr(t) = \widehat{\kappa}_r\,(z(t) - r(t))\,dt + \sigma_r\,d\widehat{W}_1(t) \tag{2.12a}$$

$$dz(t) = \widehat{\kappa}_z\,(\widehat{\theta}_z - z(t))\,dt + \sigma_z\,d\widehat{W}_2(t). \tag{2.12b}$$

This is an exponential-affine model of the term structure since $\mathcal{R}(r,z)$, $\widehat{\mu}_X(r,z)$, and $\sigma_X(r,z)\,\sigma_X(r,z)^\top$ are all affine in r and z. Therefore bond prices have the following form:

$$P(r,z,\tau) = \exp\left(-A(\tau) - B_r(\tau)\,r - B_z(\tau)\,z\right), \tag{2.13}$$

where

$$\begin{aligned} A(\tau) &= \int_{s=0}^{\tau} \widehat{\kappa}_z\,\widehat{\theta}_z\,B_z(s) - \frac{1}{2}\left(\sigma_r^2\,B_r(s) + \sigma_z^2\,B_z(s)\right) ds \\ B_r(\tau) &= \frac{1 - e^{-\widehat{\kappa}_r\,\tau}}{\widehat{\kappa}_r} \\ B_z(\tau) &= \frac{\widehat{\kappa}_z\,(1 - e^{-\widehat{\kappa}_r\,\tau}) - \widehat{\kappa}_r\,(1 - e^{-\widehat{\kappa}_z\,\tau})}{\widehat{\kappa}_z\,(\widehat{\kappa}_z - \widehat{\kappa}_r)}. \end{aligned}$$

Note that the convexity term (due to Jensen's inequality),

$$-\frac{1}{2}\left(\sigma_r^2\,B_r(\tau) + \sigma_z^2\,B_z(\tau)\right),$$

is deterministic and impounded in the constant term. It plays no role in the failure of the weak form of the expectations hypothesis in this model.

Zero-coupon yields are given by $y(t,T) = Y(r(t), z(t), T-t)$ where

$$Y(r,z,\tau) := \frac{-\log(P(r,z,\tau))}{\tau} = \widetilde{A}(\tau) + \widetilde{B}_r(\tau)\,r + \widetilde{B}_z(\tau)\,z, \tag{2.15}$$

where

$$\widetilde{A}(\tau) := \frac{A(\tau)}{\tau}, \quad \widetilde{B}_r(\tau) := \frac{B_r(\tau)}{\tau}, \quad \text{and} \quad \widetilde{B}_z(\tau) := \frac{B_z(\tau)}{\tau}.$$

Forward rates are given by $f(t,T) = F(r(t), z(t), T-t)$, where

$$F(r,z,\tau) := \frac{-\partial\,\log(P(r,z,\tau))}{\partial\,\tau} = A'(\tau) + B_r'(\tau)\,r + B_z'(\tau)\,z, \tag{2.16}$$

where

$$\begin{aligned} A'(\tau) &= \widehat{\kappa}_z\, \widehat{\theta}_z\, B_z(\tau) - \frac{1}{2}\left(\sigma_r^2\, B_r(\tau) + \sigma_z^2\, B_z(\tau)\right) \\ B_r'(\tau) &= e^{-\widehat{\kappa}_r\,\tau} \\ B_z'(\tau) &= \frac{\widehat{\kappa}_r\,(e^{-\widehat{\kappa}_r\,\tau} - e^{-\widehat{\kappa}_z\,\tau})}{\widehat{\kappa}_z - \widehat{\kappa}_r}. \end{aligned}$$

Since $r(t) = y(t,t) = f(t,t)$, we have $\mathcal{R}(r,z) = Y(r,z,0) = F(r,z,0) = r$, so that $\widetilde{A}(0) = A'(0) = 0$, $\widetilde{B}_z(0) = B_z'(0) = 0$, and $\widetilde{B}_r(0) = B_r'(0) = 1$.

One measure of the term premium is the difference between the yield to maturity on a zero-coupon bond and the average expected short-term interest rate over the same horizon:

$$\xi(r(t), z(t), \tau) := Y(r(t), z(t), \tau) - \frac{1}{\tau}\, E_t\left[\int_{s=t}^{t+\tau} r(s)\, ds\right]. \tag{2.18}$$

Integrating the conditional expectation of the interest rate given in (2.9), we have

$$\frac{1}{\tau}\, E_t\left[\int_{s=t}^{t+\tau} r(s)\, ds\right] = \left(\frac{1 - e^{-\kappa_r\,\tau}}{\kappa_r\,\tau} + 1\right)\theta_r + \left(\frac{1 - e^{-\kappa_r\,\tau}}{\kappa_r\,\tau}\right) r(t).$$

Inserting (2.15) and (2.9) into (2.18), we have

$$\begin{aligned} \xi(r,z,\tau) = \left\{\widetilde{A}(\tau) - \left(\frac{1 - e^{-\kappa_r\,\tau}}{\kappa_r\,\tau} + 1\right)\theta_r\right\} + \\ \left\{\frac{1 - e^{-\widehat{\kappa}_r\,\tau}}{\widehat{\kappa}_r\,\tau} - \frac{1 - e^{-\kappa_r\,\tau}}{\kappa_r\,\tau}\right\} r + \widetilde{B}_z(\tau)\, z. \end{aligned}$$

We can make the premium independent of r by setting $\widehat{\kappa}_r = \kappa_r$ without changing the character of the model.

Holding period returns. Define the holding period return over the period from t to $t + \delta$ on a bond that matures at time T as

$$h(t, T, \delta) := \frac{\log(p(t+\delta, T)) - \log(p(t,T))}{\delta}. \tag{2.19}$$

In terms of forward rates, we can write (2.19) as

$$\delta\, h(t,T,\delta) = \int_{s=t}^{t+\delta} f(t,s)\, ds + \int_{s=t+\delta}^{T} f(t,s) - f(t+\delta, s)\, ds, \tag{2.20}$$

where

$$f(t,T) := \frac{-\partial\, \log(p(t,T))}{\partial T}.$$

Under the expectations hypothesis, forward rates are martingales, and thus the conditional expectation of the second term on the right-hand side of (2.20) is identically zero. In this case, the expected holding period return on all bonds is the same.

Letting $P(x,\tau)$ be the function from state variable and maturity ($\tau = T - t$) to bond prices, we can write the numerator of (2.19) in terms of the general exponential-affine model as

$$\begin{aligned}\{\log(P(x(t),\tau-\delta)) - \log(P(x(t),\tau))\} + \\ \{\log(P(x(t+\delta),\tau-\delta)) - \log(P(x(t),\tau-\delta))\}\,. \end{aligned} \tag{2.21}$$

The first term in (2.21) captures the effect of reducing the maturity and depends only on the forward rate curve at time t. The second term captures the effect of changing the state variables and depends on how the state variables actually change over the holding period. Taking the unconditional expectation of (2.21) leaves only the first term, $\log(P(\theta,\tau-\delta)) - \log(P(\theta,\tau))$, since the factors are not expected to change at there unconditional means. Conditional expected holding period returns can be calculated using the conditional expectations of the state variables.

Using (2.9) and (2.13), we can write the (conditional) expected holding period return in terms of the factor loadings and state variables as

$$H(r,z,\tau,\delta) := H_0(\tau,\delta) + H_r(\tau,\delta)\,r + H_z(\tau,\delta)\,z, \tag{2.22}$$

where

$$\begin{aligned} H_0(\tau,\delta) &:= \frac{1}{\delta}\Big\{A(\tau) - A(\tau-\delta) + \\ &\qquad (e^{-\kappa_r\,\delta} - 1)\,B_r(\tau-\delta)\,\theta_r + (e^{-\kappa_z\,\delta} - 1)\,B_z(\tau-\delta)\,\theta_z\Big\} \\ H_r(\tau,\delta) &:= \frac{1}{\delta}\Big\{B_r(\tau) - e^{-\kappa_r\,\delta}\,B_r(\tau-\delta)\Big\} \\ H_z(\tau,\delta) &:= \frac{1}{\delta}\Big\{B_z(\tau) - e^{-\kappa_z\,\delta}\,B_z(\tau-\delta)\Big\}\,. \end{aligned}$$

Note that for $\tau = \delta$, the expected holding-period return is simply the yield to maturity for the bond: $H(r,z,\delta,\delta) = Y(r,z,\delta)$. We can measure *excess* expected holding-period premium as follows: $H(r,z,\tau,\delta) - Y(r,z,\delta)$. The factor loading on the interest rate for excess expected returns is given by

$$\psi_r(\tau,\delta) := \frac{B_r(\tau) - B_r(\delta) - e^{-\kappa_r\,\delta}\,B_r(\tau-\delta)}{\delta}.$$

Note that for $\tau > \delta$, $\operatorname{sign}[\psi_r(\tau,\delta)] = \operatorname{sign}[\kappa_r - \hat{\kappa}_r]$. The empirical finding that excess expected holding period returns is increasing in the interest rate requires $\kappa_r > \hat{\kappa}_r$.[2]

3. Forward rates and the future short rate

Consider the forward rate as a predictor of the future short-term interest rate. The relationship fits the classical errors-in-variables model. Since the short rate is Markovian, its conditional expectation depends only on its current value, $r(t)$. But forward rates depend on both $r(t)$ and the independent variable $z(t)$.

[2]See Hooker (1997) for an empirical investigation of holding-period returns.

The theoretical regression coefficient equals

$$\frac{\mathrm{Cov}[r(t+\tau),\, f(t,t+\tau)]}{\mathrm{Var}[f(t,t+\tau)]},$$

where $\mathrm{Var}[\,\cdot\,]$ and $\mathrm{Cov}[\,\cdot\,,\,\cdot\,]$ are the unconditional variance and unconditional covariance operators. Let's look at the numerator first:

$$\begin{aligned}\mathrm{Cov}[r(t+\tau),\, f(t,t+\tau)] &= \mathrm{Cov}[E_t[r(t+\tau)],\, f(t,t+\tau)] \\ &= e^{-(\kappa_r+\widehat{\kappa}_r)\,\tau}\ \mathrm{Var}[r(t)] \\ &= e^{-(\kappa_r+\widehat{\kappa}_r)\,\tau}\left(\frac{\sigma_r^2}{2\,\kappa_r}\right).\end{aligned}$$

Now let's look at the denominator:

$$\begin{aligned}\mathrm{Var}[f(t,t+\tau)] &= e^{-2\,\widehat{\kappa}_r\,\tau}\ \mathrm{Var}[r(t)] + \frac{\widehat{\kappa}_r^2\,(e^{-\widehat{\kappa}_r\,\tau} - e^{-\widehat{\kappa}_z\,\tau})^2}{(\widehat{\kappa}_z - \widehat{\kappa}_r)^2}\,\mathrm{Var}[z(t)] \\ &= e^{-2\,\widehat{\kappa}_r\,\tau}\left(\frac{\sigma_r^2}{2\,\kappa_r}\right) + \frac{\widehat{\kappa}_r^2\,(e^{-\widehat{\kappa}_r\,\tau} - e^{-\widehat{\kappa}_z\,\tau})^2}{(\widehat{\kappa}_z - \widehat{\kappa}_r)^2}\left(\frac{\sigma_z^2}{2\,\kappa_z}\right).\end{aligned}$$

Putting these together and rearranging a bit, we get

$$\beta(\tau) = \frac{e^{(\widehat{\kappa}_r-\kappa_r)\,\tau}}{1 + \left(\dfrac{\kappa_r\,\sigma_z^2}{\kappa_z\,\sigma_r^2}\right)\dfrac{\widehat{\kappa}_z^2\,e^{2\,\widehat{\kappa}_r\,\tau}\,(e^{-\widehat{\kappa}_r\,\tau} - e^{-\widehat{\kappa}_z\,\tau})}{(\widehat{\kappa}_r - \widehat{\kappa}_z)^2}}. \tag{3.1}$$

Note that $\beta(0) = 1$. The expectations hypothesis implies that $\beta(\tau) = 1$ for all τ. There are two channels through which the expectations hypothesis gets violated in (3.1). First, in the numerator of (3.1), $\widehat{\kappa}_r \neq \kappa_r$. Second, and more importantly, in the denominator of (3.1) the second factor is present. (If we constrain the price of z–risk to be zero, (3.1) remains correct after changing $\widehat{\kappa}_z$ to κ_z. In this case we are free to let σ_z go to zero, which makes z deterministic and eliminates the second channel.) The standard way of modeling the price of risk in a term structure model with Gaussian state variables eliminates both channels.

4. Campbell–Shiller regressions

Campbell and Shiller (1991, CS hereafter) estimated two sets of regressions involving the yield spreads. These tests of the expectations hypothesis are somewhat indirect.

The first set of regressions. Let $\beta_0(\tau-t, T-t)$ and $\beta_1(\tau-t, T-t)$ be regression coefficients indexed by $\tau - t$ and $T - t$ for $t < \tau < T$. In the first set of regressions, the change in a zero coupon yield is regressed on the slope of the yield curve:

$$y(\tau,T) - y(t,T) = \beta_0(\tau-t,T-t) + \beta_1(\tau-t,T-t)\,s(t,\tau,T) + \nu(t,\tau,T), \tag{4.1}$$

where

$$s(t,\tau,T) := \left(\frac{\tau-t}{T-\tau}\right)\Big(y(t,T) - y(t,\tau)\Big)$$

is the (weighted) slope of the yield curve.

We can take the same approach as before. We need to find

$$\beta_1(\tau - t,\, T - t) = \frac{\operatorname{Cov}[y(\tau,T) - y(t,T),\, s(t,\tau,T)]}{\operatorname{Var}[s(t,\tau,T)]}.$$

Let's start by thinking about the numerator:

$$\operatorname{Cov}[y(\tau,T) - y(t,T),\, s(t,\tau,T)] = \left(\frac{\tau - t}{T - \tau}\right) \operatorname{Cov}[E_t[y(\tau,T)] - y(t,T),\, y(t,T) - y(t,\tau)]. \quad (4.2)$$

If changes in the slope of the yield (changes in $y(t,T) - y(t,\tau)$) are driven largely by changes in the risk premia, and if those risk premia are mean reverting, then $E_t[y(\tau,T)]$ will not move much in response to a change in the risk premia, and the covariance on the right-hand side of (4.2) will be negative. In the current model, the price of risk is driven by z. An increase in z leads to an increase in the slope of the curve. This can be seen from the factor loading for z for zero-coupon bonds, which equals zero at $\tau = 0$ and increases. Since z moves independently of r, and reverts to its mean, an increase in z leads to an increase in the slope of the curve but does not increase expected future yields by as much. The effect z has on the yield curve (either forward rates or zero-coupon rates) relative to r diminishes as the maturity goes to zero. In the limit, of course, z has no effect.

Fisher and Gilles (1996b) derive expressions for the regression coefficients for generic exponential-affine models. Let

$$B(\tau) = \begin{pmatrix} B_r(\tau) \\ B_z(\tau) \end{pmatrix} \quad \text{and} \quad \Phi(\tau) = \begin{pmatrix} e^{-\kappa_r \tau} & 0 \\ 0 & e^{-\kappa_z \tau} \end{pmatrix}.$$

The regression coefficients can be written as follows (suppressing the arguments $\tau - t$ and $T - t$):

$$\beta_0 = \pi_0 - b_1\, \rho_0 + (\pi_1 - b_1\, \rho_1)\, \theta \quad (4.3a)$$

$$\beta_1 = 1 + b_1 = 1 + \frac{\rho_1\, \mathcal{V}\, \pi_1}{\rho_1\, \mathcal{V}\, \rho_1}, \quad (4.3b)$$

where

$$\rho_0(\tau - t, T - t) = \left(\frac{1}{T - \tau}\right) \left(\left(\frac{\tau - t}{T - t}\right) A(T - t) - A(\tau - t) \right)$$

$$\rho_1(\tau - t, T - t) = \left(\frac{1}{T - \tau}\right) \left(\left(\frac{\tau - t}{T - t}\right) B(T - t) - B(\tau - t) \right)$$

and

$$\pi_0(\tau - t, T - t) = \left(\frac{1}{T - \tau}\right) \Big(A(T - \tau) + A(\tau - t) - A(T - t) + B(T - \tau)^\top (I - \Phi(\tau - t))\, \theta \Big)$$

$$\pi_1(\tau - t, T - t) = \left(\frac{1}{T - \tau}\right) \Big(\Phi(\tau - t)^\top B(T - \tau) + B(\tau - t) - B(T - t) \Big).$$

For the weak form to hold, we need $\mathcal{V}\,\pi_1 = 0$. As before, there are two channels through which the weak form of the expectations hypothesis can fail.

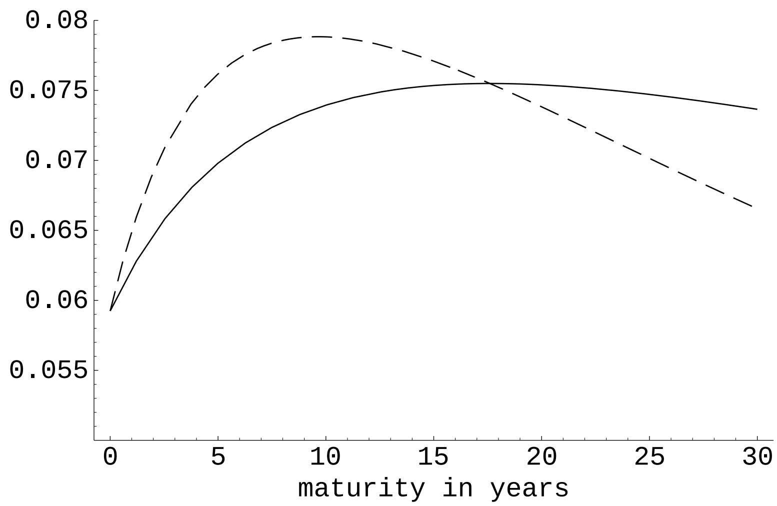

FIGURE 1. The average zero-coupon curve (solid) and forward rate curve (dashed).

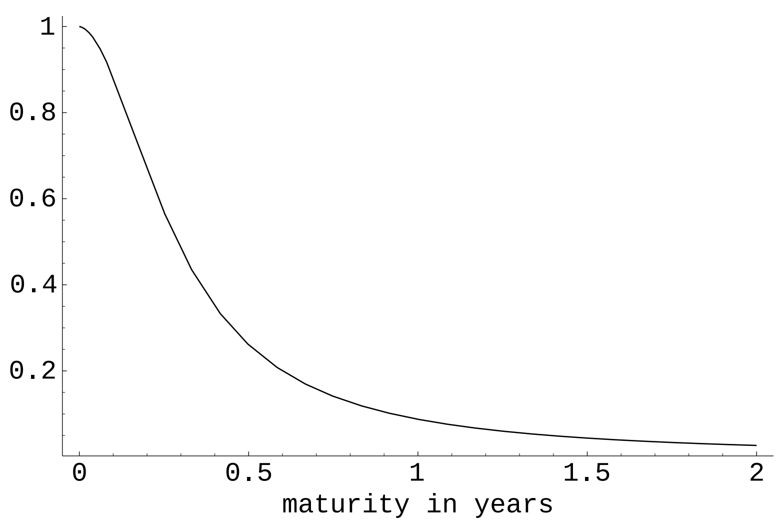

FIGURE 2. $\beta(\tau)$ in the regression of the future short-term interest rate on the current forward rate.

The second set of regressions. Let $\Delta := (T - t)/n$. CS regress what they call the "perfect-foresight spread,"

$$\frac{1}{n}\sum_{i=1}^{n-1}\bigl\{y(t+i\,\Delta,\, t+(i+1)\,\Delta) - y(t,\, t+\Delta)\bigr\},$$

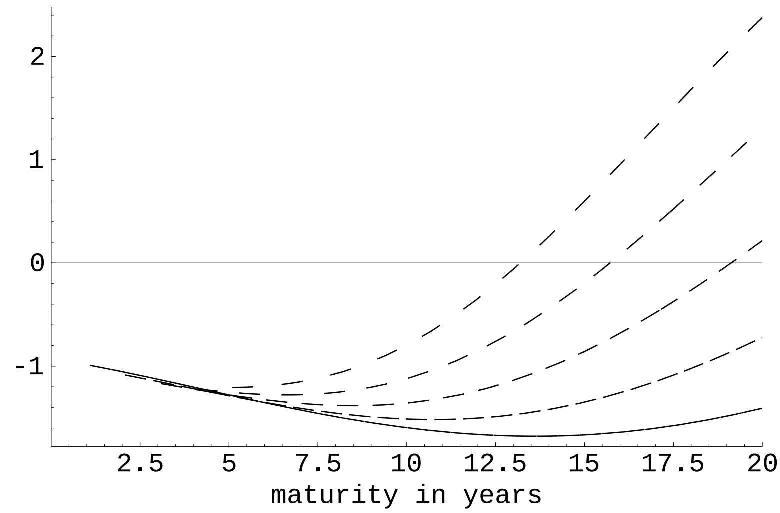

FIGURE 3. Campbell–Shiller slope coefficient $\beta(\tau_1, \tau_2)$ for $\tau_1 = 1, \ldots, 5$ and $\tau_2 = [\tau_1 + 1/12, 20]$.

on a measure of the slope of the yield curve,

$$y(t,T) - y(t, t+\Delta).$$

For simplicity, I will take the limit as $\Delta \to 0$, producing:

$$\frac{\int_{s=t}^{T} r(s) - r(t)\, ds}{T - t} \quad \text{and} \quad y(t,T) - r(t).$$

Adding $r(t)$ to each of these expressions and multiplying by $T - t$ produces

$$\int_{s=t}^{T} r(s)\, ds \quad \text{and} \quad \int_{s=t}^{T} f(t,s)\, ds.$$

Using expressions for $E_t[r(s)]$ and $f(t,s)$, we can calculate the asymptotic regression coefficients for these regressions as well.

5. ESTIMATES

I estimated the model using the maximum likelihood technique developed by Chen and Scott (1993) and described in Fisher and Gilles (1996a). The data are end-of-month zero-coupon yields from December 1988 to November 1997, extracted from coupon bond prices using the technique described in Fisher, Nychka, and Zervos (1995). I chose the 1- an 10-year yields to be measured without error and the 2-, 3-, and 30-year yields to be measured with an *ad hoc* measurement error.[3]

[3]Since the state variables are Gaussian, the model can be estimated consistently with the Kalman filter (which is surely a more aesthetically appealing technique that the one used here). I plan to do so in the future.

θ_r	θ_z	κ_r	κ_z	σ_r	σ_z	$\widehat{\theta}_z$	$\widehat{\kappa}_r$	$\widehat{\kappa}_z$
0.059	0.202	0.217	0.228	0.010	0.099	0.107	0.046	0.376
(0.009)	(0.101)	(0.157)	(0.148)	(0.001)	(0.020)	(0.007)	(0.006)	(0.049)

TABLE 1. Estimated structural parameters (and asymptotic standard errors).

Table 1 shows the parameter estimates for the model.[4] From the asymptotic standard errors (in the second row), it is evident that θ_z, κ_r, and κ_z are not well-measured by the data.

The average zero-coupon and forward rate yield curves are shown in Figure 1. The coefficient $\beta(\tau)$ is plotted in Figure 2. The Campbell–Shiller slope coefficients are shown in Figure 3.

APPENDIX A. THE STATE–PRICE DEFLATOR

In this appendix, I present the state–price deflator that produces the interest rate and price of risk in the model. The state–price deflator is given by $n(t) = \zeta(t)/u(t)$ where u is the value of a bubble asset and ζ is an exponential martingale. In particualr,

$$u(t) = \exp\left(\alpha\, t + \beta_1\, r(t) + \beta_2\, z(t)\right)$$
$$\frac{d\zeta(t)}{\zeta(t)} = -g(r(t), z(t))^\top dW(t)$$

with $\zeta(0) = 1$, where

$$\alpha = \hat{\theta}_z - \frac{1}{2}\left(\frac{\sigma_r^2}{\hat{\kappa}_r} + \frac{\sigma_z^2}{\hat{\kappa}_z}\right), \quad \beta_1 = -1/\hat{\kappa}_r \quad \text{and} \quad \beta_2 = -1/\hat{\kappa}_z,$$

and

$$g(r, z) = \mathcal{L}(r, z) + \begin{pmatrix} \sigma_r/\hat{\kappa}_r \\ \sigma_r/\hat{\kappa}_r \end{pmatrix}.$$

By applying Ito's lemma to $n(t)$, one can show that the interest rate is $r(t)$ and the price of risk is $\lambda(t) = \mathcal{L}(r(t), z(t))$.[5]

[4]The estimates of the measurement-error parameters are given in the table below, where σ_ε, ρ_c and ρ_t refer to the standard deviation of the measurement errors, their cross-sectional autocorrelation, and their serial autocorrelation. (The asymptotic standard errors are in the second row.)

σ_ε	ρ_c	ρ_t
0.0007	0.290	0.708
(0.0000)	(0.059)	(0.031)

[5]This model has no neutrino factor (*i.e.*, the neutrino factor is constant). See Fisher and Gilles (2000).

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