THE TERM STRUCTURE OF TAX-EXEMPT SPREADS: THE EFFECT OF CONVEXITY

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ABSTRACT. When taxable yields are tied to tax-exempt yields at the short end of the yield curve, the volatility of the before-tax short-term interest rate will be greater than the volatility of the after-tax short-term interest rate, and the relative shapes of the two term structures will be affected by this relationship. In particular, the yield on a default-free tax-exempt zero-coupon bond will be greater than the "after-tax yield" on a default-free taxable zero-coupon bond. Estimates of the size and variability of this effect are provided.

1. INTRODUCTION

We analyze a simple, stylized model of taxable and tax-exempt bonds in an arbitrage-free setting and derive the relationship between taxable and tax-exempt yields. The tax rule is this: Any change in the price of a taxable bond be immediately taxable as income at the fixed marginal tax rate. We focus on the yields on default-free zero-coupon bonds. The *naive hypothesis* is that the ratio of the tax-exempt yield the taxable yield equals one minus the tax rate (for all maturities). In other words, the two yield curves are proportional. We show that the naive hypothesis implies arbitrage opportunities.

The naive view does contain a germ of truth that can be used to cultivate the correct relationship. At the very short end of the term structure, the ratio of the tax-exempt spot rate to the taxable spot rate does equal one minus the tax rate. This implies that the taxable spot rate is more volatile than the tax-exempt spot rate, and the relative shapes of the two term structures will be affected by this relationship. In particular, the yield on a default-free tax-exempt zero-coupon

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This paper confirms a conjecture by Joel Lander. In terms of the notation introduced in this paper, his conjecture can be stated as follows: When $\hat{r}(t) = \phi\, r(t)$ where $\phi > 1$, then for $\tau > t$ Jensen's inequality produces $\hat{y}(t,\tau) < \phi\, y(t,\tau)$. I thank Christian Gilles and Joel Lander for useful discussions. The views expressed herein are the author's and do not necessarily reflect those of the Board of Governors of the Federal Reserve System.

bond will be greater than the "after-tax yield" on a default-free taxable zero-coupon bond.

It is useful to relate this paper to Green (1993). Green makes five assumptions (p. 327):

1. There are taxable and tax-exempt bonds priced at par available for each maturity.
2. All bonds are riskless and noncallable.
3. [a] Bonds are priced on the basis of cash flows generated if held to maturity.
   [b] In particular, the options to realize capital losses early are ignored.
4. Investors trade freely and without frictions in all markets simultaneously, and taxation on long and short positions is completely symmetric so that investors are "marginal" on all bonds.
5. Over the life of each bond the tax rate and the tax regime will not change.

I adopt these assumptions with the expectation of Assumption 3, each part of which deserves separate comment. Regarding part [b], the tax-system treated in this paper does not allow for the deferral of capital gains taxes and consequently, there are no opportunities for realizing capital losses early. By contrast, Green's treatment of the tax code is significantly more realistic. In fact, his analysis is based on the possibility of constructing a taxable zero-coupon bond for which the taxes are deferred until maturity.[1] Regardless of the tax system adopted, part [a] of Assumption 3 is at odds with standard absence-of-arbitrage conditions that effectively take into account short term fluctuations in asset prices. These play a central role in this paper.

In Section 2 we use the Heath, Jarrow, and Morton (1992) restriction on the dynamics for forward rates to show that two yield curves cannot be proportional absent arbitrage opportunities. In Section 3 we show that, except for instantaneous spot rates, the ratio of tax-exempt yields to taxable yields must be greater than that directly implied by the tax rate. In Section 4 we show how to take an exponential-affine model of the taxable term structure and derive the tax-exempt term structure. We provide empirical examples based on three models estimated by Chen and Scott (1993). In Section 5 we discuss how to incorporate default risk into the tax-exempt term structure.

2. The naive hypothesis

There are two types of zero-coupon bonds: *tax-exempt* and *taxable.* Both types of bonds pay one unit at maturity with certainty. The taxable bonds are taxable in the following way: Any change in price is immediately taxable as income at the fixed marginal tax rate ξ, where $0 < \xi < 1$. Let the price at time t of a tax-exempt

[1]Green also recognizes that "as the tax code in fact stipulates, that interest deductions from borrowings and short positions cannot exceed investment income from the positions these borrowings support . . . ."

bond that matures at τ be $P(t,\tau)$, and let the price at time t of a taxable bond that matures at τ be $\hat{P}(t,\tau)$. The yield-to-maturity and instantaneous forward rates are given by

$$\begin{aligned} y(t,\tau) &:= \frac{-\log(P(t,\tau))}{(\tau-t)} \\ f(t,\tau) &:= -\frac{\partial \log(P(t,\tau))}{\partial \tau} = y(t,\tau) + (\tau - t)\,\frac{\partial\, y(t,\tau)}{\partial \tau} \end{aligned}$$

for tax-exempt bonds, with analogous expressions for taxable bonds.

The naive hypothesis is $\hat{y}(t,\tau) = \phi\, y(t,\tau)$, where $\phi := 1/(1-\xi) > 1$. This hypothesis can also be written in terms of either forward rates, $\hat{f}(t,\tau) = \phi\, f(t,\tau)$, or bond prices, $\hat{P}(t,\tau) = P(t,\tau)^{\phi}$. The most immediate way to see that the naive hypothesis cannot hold absent arbitrage opportunities is to apply the Heath, Jarrow, and Morton (1992, HJM) absence-of-arbitrage conditions for forward-rate dynamics. Following HJM, assume that $f(t,\tau)$ can be written as an Ito process under the standard equivalent martingale measure:

$$df(t,\tau) = \mu_f(t,\tau)\,dt + \sigma_f(t,\tau)^{\top} dW(t).$$

As HJM first showed, the absence of arbitrage implies

$$\mu_f(t,\tau) = \sigma_f(t,\tau)^{\top} \int_t^{\tau} \sigma_f(t,u)\,du. \tag{2.1}$$

Given the naive hypothesis, $\hat{f}(t,\tau) = \phi\, f(t,\tau)$, we have $d\hat{f}(t,\tau) = \phi\, df(t,\tau)$, and hence we can write

$$d\hat{f}(t,\tau) = \hat{\mu}_f(t,\tau)\,dt + \hat{\sigma}_f(t,\tau)^{\top} dW(t),$$

where

$$\begin{aligned} \hat{\mu}_f(t,\tau) &= \phi\, \mu_f(t,\tau) \\ \hat{\sigma}_f(t,\tau) &= \phi\, \sigma_f(t,\tau). \end{aligned} \tag{2.2}$$

But there is an analogous HJM restriction for the dynamics of $\hat{f}(t,\tau)$ to contend with:

$$\hat{\mu}_f(t,\tau) = \hat{\sigma}_f(t,\tau)^{\top} \int_t^{\tau} \hat{\sigma}_f(t,u)\,du. \tag{2.3}$$

Substituting (2.2) into (2.3) produces (after cancellation)

$$\mu_f(t,\tau) = \phi\, \sigma_f(t,\tau)^{\top} \int_t^{\tau} \sigma_f(t,u)\,du. \tag{2.4}$$

Obviously (2.4) contradicts (2.1), and therefore, it cannot be the case that $\hat{y}(t,\tau) = \phi\, y(t,\tau)$ absent arbitrage.

3. The correct relationship

Having ruled out $\hat{y}(t,\tau) = \phi\, y(t,\tau)$, can we say anything stronger about the correct relationship between $\hat{y}(t,\tau)$ and $y(t,\tau)$? The answer is yes. The line of reasoning we follow is this: We show that (under the standard equivalent martingale measure $\mathcal{Q}$) the expected rate of return on the tax-exempt bond is $r(t)$ and the before-tax expected rate of return on the taxable bond is $\hat{r}(t) := \phi\, r(t)$. This allows us to express tax-exempt and taxable bond prices as $P(t,\tau) = E_t[x]$ and $\hat{P}(t,\tau) = E_t[x^\phi]$, where x is a time-τ measurable random variable. Jensen's inequality then implies $\hat{y}(t,\tau) < \phi\, y(t,\tau)$.

Let the tax-exempt short rate be $r(t)$. (Note that $r(t) \equiv y(t,t)$.) The value of the tax-exempt money-market account at time t is

$$\beta(t) := \exp\left(\int_0^t r(u)\, du\right).$$

Under $\mathcal{Q}$, $P(t,\tau)/\beta(t)$ is a martingale. Therefore we can write

$$\frac{dP(t,\tau)}{P(t,\tau)} = r(t)\, dt + \sigma_P(t,\tau)^\top dW(t) \tag{3.1}$$

and

$$P(t,\tau) = E_t\left[\beta(t)/\beta(\tau)\right] = E_t\left[\psi(t,\tau)\right], \tag{3.2}$$

where $\psi(t,\tau) = \beta(t)/\beta(\tau) = \exp\left(-\int_t^\tau r(u)\, du\right)$.

Now let us turn to taxable bonds.[2] Given the tax rule, the dynamics of the tax flow are $\xi\, d\hat{P}(t,\tau)$. Therefore the after-tax return on a taxable bond is given by $(1-\xi)\, d\hat{P}(t,\tau)/\hat{P}(t,\tau)$. Under $\mathcal{Q}$ the expected after-tax rate of return must equal the tax-exempt spot rate: $E_t[(1-\xi)\, d\hat{P}(t,\tau)/\hat{P}(t,\tau)] = r(t)\, dt$ or $E_t[d\hat{P}(t,\tau)/\hat{P}(t,\tau)] = \hat{r}(t)\, dt$, where $\hat{r}(t) := \phi\, r(t)$. Therefore we can write

$$\frac{d\hat{P}(t,\tau)}{\hat{P}(t,\tau)} = \hat{r}(t)\, dt + \hat{\sigma}_P(t,\tau)^\top dW(t). \tag{3.3}$$

Before proceeding, we can see again that the naive hypothesis violates the absence of arbitrage: Ito's lemma applied to $P(t,\tau)^\phi$ delivers

$$\frac{dP(t,\tau)^\phi}{P(t,\tau)^\phi} = \left(\hat{r}(t) + \xi\, \frac{1}{2}\|\phi\, \sigma_P(t,\tau)\|^2\right) dt + \phi\, \sigma_P(t,\tau)^\top dW(t). \tag{3.4}$$

Clearly the drift in (3.4) does not equal the drift in (3.3), and therefore we cannot have $\hat{y}(t,\tau) = \phi\, y(t,\tau)$ absent arbitrage.

To find an expression for taxable bonds analogous to (3.2), define

$$\hat{\beta}(t) := \exp\left(\int_0^t \hat{r}(s)\, ds\right) = \beta(t)^\phi,$$

[2] In the Appendix we present a more formal derivation of (3.3).

and note that $\hat{P}(t,\tau)/\hat{\beta}(t)$ is a martingale under $\mathcal{Q}$. Therefore the price of a taxable bond can be written as

$$\hat{P}(t,\tau) = E_t\left[\hat{\beta}(t)/\hat{\beta}(\tau)\right] = E_t\left[\psi(t,\tau)^{\phi}\right]. \tag{3.5}$$

We can examine the relationship between $y(t,\tau)$ and $\hat{y}(t,\tau)$ by comparing (3.2) and (3.5). First, assuming the interest rate must remain positive (so that $\phi(t,\tau) < 1$ for $\tau > t$), we have $P(t,\tau) > \hat{P}(t,\tau)$, and hence $\hat{y}(t,\tau) > y(t,\tau)$.[3] Second, since x^{ϕ} is convex in x when $\phi > 1$, Jensen's inequality says $E[x]^{\phi} < E[x^{\phi}]$. Thus, for $\tau > t$

$$P(t,\tau)^{\phi} < \hat{P}(t,\tau).$$

Taking logs and rearranging produces the central result: For $\tau > t$

$$\hat{y}(t,\tau) < \phi\, y(t,\tau). \tag{3.6}$$

Thus for $\tau > t$ the absence of arbitrage requires that the ratio of tax-exempt to taxable yield satisfy

$$1 - \xi < \frac{y(t,\tau)}{\hat{y}(t,\tau)} < 1. \tag{3.7}$$

4. Modeling the term structure

Sharper statements about the relationship between $\hat{y}(t,\tau)$ and $y(t,\tau)$ require an explicit model of the term structure. Consider the exponential-affine class of models of the term structure of interest rates.[4] In this class of models, we have $r(t) = \delta_0 + \sum_{i=1}^{d} \delta_i\, X_i(t)$, where the $X_i(t)$ are Markovian state variables. The vector of state variables $X(t)$ has the following dynamics under the standard equivalent martingale measure:

$$dX(t) = \mu_X(X(t))\, dt + \sigma_X(X(t))^{\top} dW(t),$$

where

$$\mu_X(x) = b_0 + \sum_{i=1}^{d} b_i\, x_i, \tag{4.1a}$$

$$\sigma_X(x)^{\top}\sigma_X(x) = G_0 + \sum_{i=1}^{d} G_i\, x_i, \tag{4.1b}$$

where the δ_i are scalars, the b_i are $d\times 1$ vectors and the G_i are $d\times d$ matrices. Abusing notation slightly, bond prices as a function of the state variables and maturity $m = \tau - t$ are given by

$$P(x,m) = \exp\left(-B_0(m) - \sum_{i=1}^{d} B_i(m)\, x_i\right),$$

[3] In a Vasicek (1977) model for example, the interest rate can be negative and one can have $P(t,\tau) < \hat{P}(t,\tau)$.

[4] See Duffie and Kan (1995) for a general discussion of this class of models. See Fisher and Gilles (1996) for a discussion of estimating these models.

where the $B_i(m)$ solve[5]

$$B_i'(m) = \delta_i + B(m)^\top b_i - \frac{1}{2}\, B(m)^\top G_i\, B(m) \tag{4.2}$$

subject to $B_i(0) = 0$ for $i = 0$ to d. The initial conditions follow from the requirement that $P(x, 0) = 1$.

For taxable bonds, the only change in the structure of the model is due to $\hat{r}(t) = \phi\, r(t)$: Thus we have $\hat{r}(t) = \hat{\delta}_0 + \sum_{i=1}^{d} \hat{\delta}_i\, X_i(t)$, where $\hat{\delta}_i = \phi\, \delta_i$, and the rest of the model is unchanged. Therefore, we can write

$$\hat{P}(x, m) = \exp\left(-\hat{B}_0(m) - \sum_{i=1}^{d} \hat{B}_i(m)\, x_i\right),$$

where the $\hat{B}_i(m)$ solve

$$\hat{B}_i'(m) = \phi\, \delta_i + \hat{B}(m)^\top b_i - \frac{1}{2}\, \hat{B}(m)^\top G_i\, \hat{B}(m) \tag{4.3}$$

subject to $\hat{B}_i(0) = 0$ for $i = 0$ to d.

Empirical section. For empirical purposes, it is convenient to take the taxable term structure as the primitive and derive the tax-exempt term structure from it. Thus, given an estimated model of the taxable term structure and a marginal tax rate ξ, we can immediately solve for the tax-exempt term structure, using $\delta_i = (1 - \xi)\, \hat{\delta}_i$.

Chen and Scott (1993), for example, provide estimates of one-, two-, and three-factor exponential-affine models of the taxable default-free term structure. In each of their models the interest rate equals the sum of the factors, $r(t) = \sum_{i=1}^{d} X_i(t)$, and the factors follow independent square-root processes under $\mathcal{Q}$:

$$dX_i(t) = (\kappa_i + q_i)\, (\theta_i - X_i(t))\, dt + \sigma_i \sqrt{X_i(t)}\, dW_i(t),$$

The q_i are related to the market price of risk and are identified by the time-series properties of the term structure under the physical measure.[6]

Figure 1 shows the ratio $y(t, \tau)/\hat{y}(t, \tau)$ for each of the three models with $\xi = 0.30$. The state variables are set at their unconditional means. Figure 2 shows the corresponding forward-rate ratios $f(t, \tau)/\hat{f}(t, \tau)$. The long-dashed line is the one-factor model; the short-dashed line is the two-factor model; and the alternating long-and-short dashed line is the three factor model. In all cases, condition (3.7) is satisfied. However, it is clear that the one-factor model is fundamentally different from the two- and three-factor models. Moreover, the one-factor model is quite close to the naive relationship. However, it has been documented that the one factor model *does not* capture the major features of the term structure in terms of its shape and dynamics.

The par-coupon rate ratio is of some interest. The par coupon rate is the coupon rate that makes a coupon bond be valued at par. For simplicity, we define the

[5]In this class of models, the PDE for bond prices decomposes into a set of ODEs.

[6]This parameterization produces $b_{0i} = (\kappa_i + q_i)\, \theta_i$, $b_{ii} = -(\kappa_i + q_i)$, $b_{ij} = 0$ for $j \neq i > 0$, $G_0 = 0$, $G_{ii} = \sigma_i^2$, and $G_{ij} = 0$ for $j \neq i > 0$.

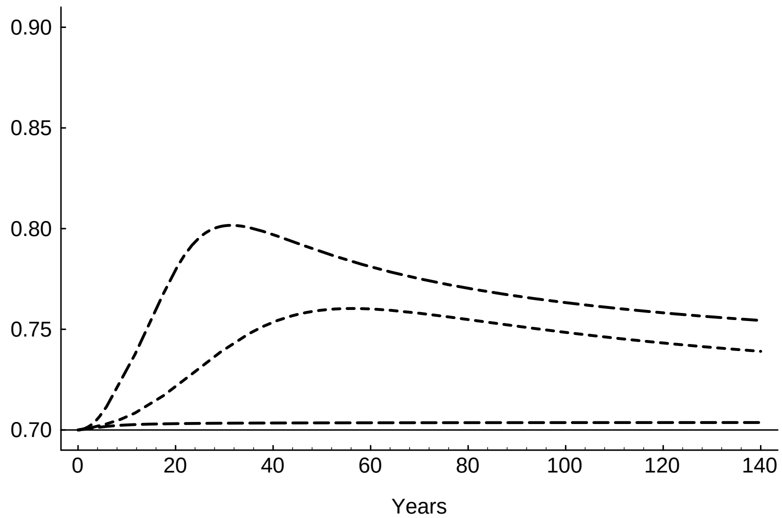

FIGURE 1. Zero curve ratios: 30 percent tax rate

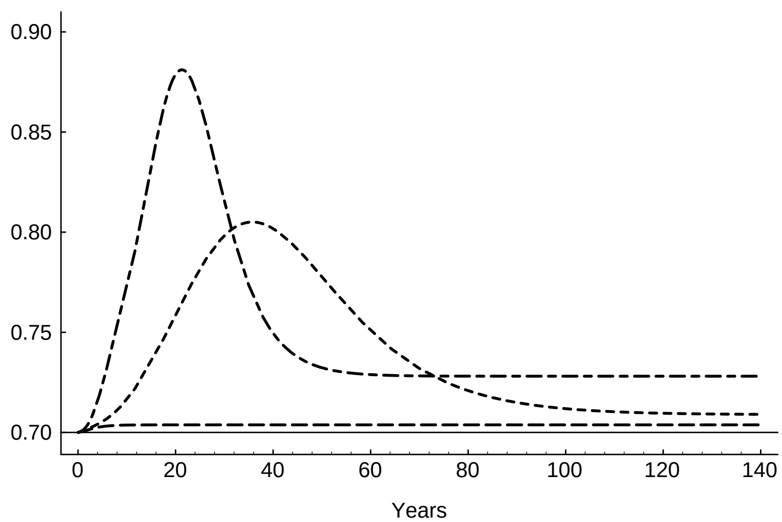

FIGURE 2. Forward curve ratios: 30 percent tax rate

par-coupon rate in terms of a bond that pays a continuous coupon at rate. The value of a continuous-coupon bond that pays coupons at rate c is

$$\int_t^\tau c\, P(t,u)\, du + P(t,\tau).$$

Therefore, the par-coupon rate (as a function of time and maturity) is

$$c(t,\tau) = \frac{1 - P(t,\tau)}{\int_t^\tau P(t,u)\, du}. \tag{4.4}$$

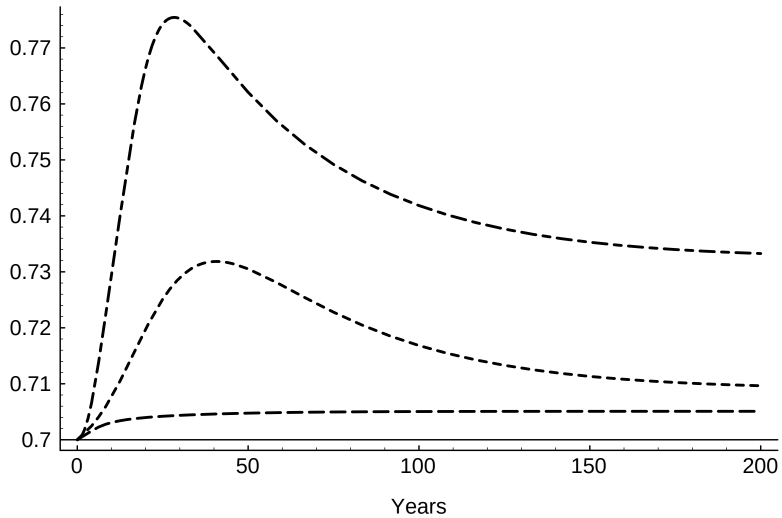

FIGURE 3. Par-coupon curve ratios: 30 percent tax rate

We define $\hat{c}(t,\tau)$ analogously. We note that we can invert this relationship:

$$P(t,\tau) = 1 - c(t,\tau) \int_t^\tau \exp\left(-\int_s^\tau c(t,u)\,du\right) ds.$$

Suppose $\hat{c}(t,\tau) = \phi\, c(t,\tau)$. Then we have

$$\hat{P}(t,\tau) = 1 - \phi\, c(t,\tau) \int_t^\tau \exp\left(-\int_s^\tau c(t,u)\,du\right)^{\phi} ds. \tag{4.5}$$

There does not appear to be a clear relationship between $\hat{P}(t,\tau)$ as defined in (4.5) and $P(t,\tau)^\phi$.

5. Time-dependent tax rates

One can easily generalize the foregoing results to a time-dependent tax rate, $\xi(t)$, where the time dependence is deterministic. The absence-of-arbitrage reasoning in Section 3 leads to $\hat{r}(t) = \phi(t)\, r(t)$, where $\phi(t) = 1/(1-\xi(t))$. In terms of Section 4, we have $\delta_i(t) = (1-\xi(t))\,\hat{\delta}_i$. We now have time entering bond prices independently of maturity: bond prices as a function of the state variables and maturity $m = \tau - t$ are given by

$$P(x,m;t) = \exp\left(-B_0(m;t) - \sum_{i=1}^d B_i(m;t)\,x_i\right),$$

where the $B_i(m;t)$ solve

$$B_i'(m;t) = \delta_i(m+t) + B(m;t)^\top b_i - \frac{1}{2}\,B(m;t)^\top G_i\,B(m;t) \tag{5.1}$$

subject to $B_i(0;t) = 0$ for $i = 0$ to d.

6. Default risk

Tax-exempt bonds, however, are typically subject to default risk.[7] A convenient way to handle default risk is to model the intensity of the risk of default (in a Poisson process sense) under $\mathcal{Q}$ as the spread between risk-free tax-exempt spot rate and the risky tax-exempt spot rate: $\tilde{r}(t) = r(t) + h(t)$, where $h(t) \geq 0$.[8] This framework allows us to can start from a default-free tax-exempt term structure and add a default factor to get a tax-exempt term structure with credit risk. The relationship between the default-free taxable spot rate and the credit-risky tax-exempt spot rate, $\tilde{r}(t)$, then is given by

$$\tilde{r}(t) = h(t) + (1 - \xi)\, \hat{r}(t).$$

At the short end of the curve, we have

$$\frac{\tilde{r}(t)}{\hat{r}(t)} = (1 - \xi) + \frac{h(t)}{\hat{r}(t)}.$$

Now consider four kinds of zero-coupon bonds: (i) tax-exempt default-free (ideal), (ii) taxable default-free (Treasury), (iii) tax-exempt defaultable (muni), and (iv) taxable defaultable (corporate). Let r_e denote the instantaneous expected rate of return on ideal bonds under the risk-neutral measure, and let r_T, r_m, and r_c denote the gross expected returns on the other three bonds. Then we have the following relations:

$$r_T = \phi_T\, r_e, \quad r_m = r_e + h, \quad \text{and} \quad r_c = \phi_c\, r_m = \phi_c\, r_e + \phi_c\, h.$$

If we let $r_T(t) = R(X(t))$ and $\phi_c\, h(t) = H(X(t))$, then we have

$$\begin{aligned} r_T(t) &= R(X(t)) \\ r_m(t) &= \frac{1}{\phi_T}\, R(X(t)) + \frac{1}{\phi_c}\, H(X(t)) \\ r_c(t) &= \frac{\phi_c}{\phi_T}\, R(X(t)) + H(X(t)). \end{aligned}$$

Appendix A.

Let V be the value of a strictly positive asset and D be the cumulative dividend process for that asset.[9] We can write the dynamics of V and D under the martingale measure as follows:

$$\begin{aligned} \frac{dV(t)}{V(t)} &= \mu_V(t)\, dt + \sigma_V(t)^\top dW(t) \\ \frac{dD(t)}{V(t)} &= \mu_D(t)\, dt + \sigma_D(t)^\top dW(t). \end{aligned}$$

[7] See Duffie and Singleton (1995) for a discussion of modeling defaultable term structures.

[8] Note that the intensity of default under the physical measure is given by $h(t)/(1 + \lambda_h(t))$, where $\lambda_h(t)$ is the market price of default risk.

[9] See Duffie (1996, Chapter 6, Section K) for a discussion of cumulative dividends and deflated gains.

The deflated gain is given by

$$G^{\beta}(t) = \frac{V(t)}{\beta(t)} + \int_0^t \frac{dD(s)}{\beta(s)}.$$

The absence of arbitrage implies that $G^{\beta}(t)$ is a martingale under the standard equivalent martingale measure $\mathcal{Q}$. Since

$$dG^{\beta}(t) = \frac{dV(t) + dD(t) - r(t)\,V(t)\,dt}{\beta(t)},$$

the absence of arbitrage implies

$$\mu_V(t) = r(t) - \mu_D(t). \tag{A.1}$$

Given the tax rule $dD(t) = -\xi\,dV(t)$, we have $\mu_D(t) = -\xi\,\mu_V(t)$, which we can insert into (A.1) and solve for

$$\mu_V(t) = \phi\,r(t),$$

where $\phi = 1/(1-\xi)$.

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