CONSUMPTION AND ASSET PRICES WITH HOMOTHETIC RECURSIVE PREFERENCES

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Abstract. When preferences are homothetic, utility can be expressed in terms of current consumption and a variable that captures all information about future opportunities. We use this observation to express the differential equation that characterizes utility as a restriction on the information variable in terms of the dynamics of consumption. We derive the supporting price system and returns process and thereby characterize optimal consumption and portfolio decisions. We provide a fast and accurate numerical solution method and illustrate its use with a number of Markovian models. In addition, we provide insight by changing the numeraire from units of consumption to units of the consumption process. In terms of the new units, the wealth–consumption ratio (which is closely related to the information variable) is the value of a coupon bond and the existence of an infinite-horizon solution depends on the positivity of the asymptotic forward rate.

1. Introduction

We solve for the dynamics of consumption, investment, and asset prices in a general-equilibrium, continuous-time stochastic model with a representative agent who has recursive preferences. The setting varies and determines the problem that we solve. In an endowment economy, the dynamics of consumption is given and we solve for asset prices (the *exchange problem*); in a partial equilibrium setting, prices are given and we solve for the optimal consumption and investment plan (the *planning problem*); in a production economy, a set of linear technologies is given and we solve for consumption, investment and asset prices (the *production problem*). By focusing on the consumption–wealth ratio, we find that these three problems are essentially equivalent, and we provide a unified framework for solving them.

The recursive utility framework generalizes the standard time-separable power utility model, allowing the separation of risk aversion and intertemporal substitution. This framework was introduced by Epstein and Zin (1989), who analyze

Date: June 4, 1999.

JEL Classification. G12.

Key words and phrases. Recursive preferences, stochastic differential utility, general equilibrium, optimal consumption, optimal portfolio, equity premium, term structure of interest rates, asset pricing.

We thank Greg Duffee and Costis Skiadas for useful conversations. The views expressed herein are the authors' and do not necessarily reflect those of the Federal Reserve Bank of Atlanta, the Federal Reserve System, or Bear, Stearns & Co. Note: This is a substantially revised version of "Consumption and asset prices with recursive preferences" that appeared as a Federal Reserve Board working paper (FEDS 98-40).

recursive preferences in a discrete-time setting, and Duffie and Epstein (1992b), who develop a continuous-time formulation of Epstein and Zin's class of recursive utility called stochastic differential utility. We use a martingale approach to solve for the equilibrium, along the lines of Duffie and Skiadas (1994), who show that the first-order condition for optimality is equivalent to the absence-of-arbitrage conditions for asset prices—namely, that asset prices deflated by the state-price deflator are martingales. In addition, they provide a representation for the state-price deflator for the Kreps–Porteus stochastic differential utility (K–P SDU) that we adopt here.

We express utility in terms of two state variables: current consumption and a second variable (the *information variable*) that captures all information about future opportunities. This representation of utility relies on the homotheticity of K–P SDU, and holds for the exchange problem as well as in economies with linear investment opportunities (covering both the case of the planning problem and that of the production problem). Equilibrium in the model reduces to a central restriction on the information variable in terms of the dynamics of a forcing process. This forcing process can be either consumption (for the exchange problem), the real state–price deflator (for the planning problem), the return on the market portfolio (for the production problem), or something entirely different (for example, the state-price deflator expressed in an arbitrary numeraire). Solving the model for (*i*) optimal consumption, (*ii*) the optimal portfolio, and (*iii*) asset prices amounts to finding the process for the growth variable that satisfies the central restriction.

Unless the elasticity of intertemporal substitution is unity, we can replace the information variable with the wealth–consumption ratio. The homogeneity properties of the representative agent's planning problem (homothetic preferences and linear technology) ensure that optimal consumption is proportional to wealth. We show that the optimal wealth–consumption ratio is the value of a coupon bond when the numeraire has been changed from units of the consumption good to shares in the consumption process (*i.e.*, the dividend process). Thus, the wealth-consumption ratio is the value of an asset. As such, it must obey a standard absence-of-arbitrage condition.

As a practical matter, the model is solved when we know how to obtain, analytically or numerically, an expression for the consumption-wealth ratio that satisfies this condition. It is then straightforward to obtain expressions for the rate of interest and the price of risk—determined by the dynamics of the so-called *state-price deflator*—and other variables of interest. In order to focus on the role of preferences, it is convenient, in the spirit of Lucas (1978) (as well as Mehra and Prescott (1985) and Weil (1989)), to start with the exchange problem, in which the forcing process is consumption and we solve for the supporting prices, *i.e.*, the state-price deflator. For the planning problem, we reverse the process, solving for the optimal consumption and investment plans using the state-price deflator as the forcing process. Finally, in the spirit of Cox, Ingersoll, Jr., and Ross (1985a) and Campbell (1993), we model technology, which we interpret as the return on the optimally invested wealth of the representative consumer. For this production problem, then, we solve for consumption and prices using the return on the market portfolio as the forcing process.

In a Markovian setting, the dynamics of the forcing process are driven by a finite set of Markovian state variables. In such a Markovian setting the no-arbitrage condition becomes a partial differential equation (PDE) that we wish to solve for the wealth-consumption ratio as a function of the state variables (and time). Because this ratio is the value of a dividend-denominated coupon bond, it can be computed from dividend-denominated bond prices. In some circumstances, standard methods deliver exact solutions (numerically at least and sometimes even analytically) to the bond pricing problem, and we get the wealth-consumption ratio by numerical integration. In all other cases, we attack the annuity PDE directly and provide an approximate solution method that produces fast and accurate numerical solutions that converge to the Taylor expansion of the exact solution. Much like standard bond pricing methods, our solution method transforms the PDE into a set of simultaneous ordinary differential equations (ODE) when the horizon is finite. A unique solution is guaranteed to exist, but only for horizons that are sufficiently short. We solve the infinite-horizon problem by extending the finite horizon and taking a limit. Such a limit does not necessarily exist, but when it does, it is the solution of a set of algebraic equations. The conditions for the existence of the solution to the infinite-horizon problem can be understood in terms of the dividend-denominated term structure. Since the wealth–consumption ratio is the value of a fixed-income security, it will be finite in the limit only if the asymptotic dividend-denominated forward rate is positive.

Related work. As noted above, Duffie and Epstein (1992b) and Duffie and Skiadas (1994) lay the groundwork for continuous-time modeling of recursive preferences. Schroder and Skiadas (1999) extend the earlier work in a number of important ways. They prove existence and uniqueness of solutions and address the relation between the first-order conditions and optimality in a more general non-Markovian setting than has been treated previously, and we refer the reader to their paper regarding these issues. In addition, they provide some closed-form solutions to the planning problem in special cases that we also consider below.

Duffie and Epstein (1992a) derive the representation for risk premia in the setting we adopt here. Both Duffie and Epstein (1992a) and Duffie, Schroder, and Skiadas (1997) solve one-factor models of the term structure in the special case where the dynamics of the state variable are introduced through the growth rate of consumption. Among other things, these papers address how a change in the coefficient of relative risk aversion affects the shape of the yield curve. Duffie and Lions (1992) address questions of existence and uniqueness of solutions to the PDE in a setting similar to ours.

Campbell (1993) linearizes the discrete-time model of Epstein and Zin (1991), and derives an approximate solution to the model in the homoskedastic case that is exact when the intertemporal rate of substitution equals unity. We derive more general conditions under which important aspects of Campbell's solution are essentially exact, providing insight into the performance of his approximate solutions.

In addition, we examine the approximate relations Campbell describes between the volatility of a perpetuity and the price of risk. Campbell's model is used by Campbell and Viceira (1996) to study the planning problem.

Outline. In Section 2, we introduce the utility function (K–P SDU), for which we derive a two-state-variable representation in the context of the exchange problem, thereby simplifying the model's central restriction. The state variables independently capture the level and growth features of the endowment process.

In Section 3, we derive the returns process that supports the endowment and we show that the wealth–consumption ratio depends only on the information variable. Next, we change perspective and characterize the solution of the optimal consumption and optimal portfolio problems.

In Section 4, we show that the wealth–consumption ratio is the value of a coupon bond after the numeraire has been changed from units of consumption good to shares in the endowment itself (*i.e.*, the dividend). Next we show how to solve the model when the dividend-denominated interest rate is deterministic or Gaussian. In the case of unit elasticity of intertemporal substitution combined with homoskedasticity, we show that the growth variable is a weighted average of expected future growth rates of the forcing variable. We show that the weak form of the expectations hypothesis as applied to the endowment term structure delivers useful results. In addition we show how to identify the boundary between regions of convergence and nonconvergence to an infinite-horizon solution using the asymptotic dividend-denominated forward rate. Finally, we examine a number of limiting cases regarding the preference parameters.

In Section 5, we introduce a Markovian structure that turns the central restriction into a partial differential equation (PDE), and we present some illustrations. In Section 6, we present our numerical solution method and illustrate it with some examples.

2. Homothetic SDU

We now introduce the preferences of the representative agent, for which we adopt Kreps–Porteus stochastic differential utility (SDU). We adopt the stochastic framework studied in Duffie (1996), to which we refer the reader for all omitted details. We restrict attention to a Brownian environment, by which we mean that we are given a l-dimensional vector of orthonormal Brownian motions, $W(t)$, defined on a fixed probability space, and the filtration is that generated by $W(t)$. In other words, the information that agents have at time t is that contained in the path of $W(s)$ for $s < t$.

We present a value function for Kreps–Porteus SDU that is valid for the entire parameter space. Using this value function and the general representation for the SDU gradient given by Duffie and Skiadas (1994), we obtain an explicit representation for the state-price deflator. We derive expressions for the interest rate and the price of risk in terms of this representation.

As explained by Duffie and Epstein (1992a) and Duffie and Epstein (1992b), SDU (not just the Kreps–Porteus specification) can be represented by a pair of functions

(f, A) called an *aggregator*. The functions f and A can be interpreted as capturing separately attitudes toward intertemporal substitution and attitudes toward risk in the following sense. First, in a deterministic setting, A plays no role. Second, hypothetical experiments can be conducted, for example, by fixing f and varying A to study the effect of risk aversion. Associated with the aggregator, there is a process $V(t)$, called *continuation utility*, such that the value of the consumption plan $\{c\} = \{c(t) \mid t \geq 0\}$, an Ito process, is $U(\{c\}) = V(0)$. Continuation utility is also an Itô process, and we can write its dynamics as

$$dV(t) = \mu_V(t)\,dt + \sigma_V(t)^\top dW(t).$$

Continuation utility has the following representation:

$$V(t) = E_t\left[\int_{s=t}^{T} f(c(s), V(s)) + A(V(s))\,\frac{1}{2}\,\|\sigma_V(s)\|^2\,ds + G(T)\right], \tag{2.1}$$

where $G(T)$ is a terminal reward and $E_t[\,\cdot\,]$ denotes the expectation given information available at time t. Equation (2.1) can represent either finite- or infinite-horizon utility, depending on how the terminal reward is modeled.[1] Applying Itô's lemma to (2.1) produces an equivalent characterization of V:

$$\mu_V(t) = -f(c(t), V(t)) - A(V(t))\,\frac{1}{2}\,\|\sigma_V(t)\|^2, \quad \text{subject to } V(T) = G(T). \tag{2.2}$$

We adopt the following standing assumption: the consumption process is positive; its dynamics are given by[2]

$$d\log(c(t)) = \widetilde{\mu}_c(t)\,dt + \sigma_c(t)^\top dW(t). \tag{2.3}$$

By definition, preferences are homothetic if $U(\{c'\}) \geq U(\{c\}) \iff U(\{\lambda\,c'\}) \geq U(\{\lambda\,c\})$ for $\lambda > 0$. If preferences are homothetic, there is a monotonic transformation of $U(\{c\})$ that is linearly homogeneous in consumption. Duffie and Epstein (1992b) show that $V(0) = U(\{c\})$ is linearly homogeneous in consumption if and only if the aggregator (f, A) satisfies (i) f is homogeneous of degree 1 and (ii) A is linearly homogeneous of degree -1 (in which case $V(t)$ is linearly homogeneous for any t). Their proof relied on the terminal condition $G(T) = 0$ but is equally valid for the linearly homogeneous terminal condition $G(T) = \zeta\,c(T)$, where $\zeta \geq 0$ is

[1] See the appendix in Duffie and Epstein (1992b) for a formal treatment of extending finite-horizon SDU to the infinite-horizon case.

[2] We use the following notational convention. If $z(t)$ is explicitly strictly positive, then μ_z, $\widetilde{\mu}_z$ and σ_z refer to the quantities implicitly defined in $dz(t)/z(t) = \mu_z(t)\,dt + \sigma_z(t)^\top dW(t)$ and $d\log(z(t)) = \widetilde{\mu}_z(t)\,dt + \sigma_z(t)^\top dW(t)$, implying $\widetilde{\mu}_z(t) := \mu_z(t) - \frac{1}{2}\,\|\sigma_z(t)\|^2$. There are two notable exceptions. One is continuation utility, $V(t)$. The other is the state variables $X(t)$ (introduced below), which are not necessarily positive. For these variables, we write $dX(t) = \mu_X(t)\,dt + \sigma_X(t)^\top dW(t)$, so that $\mu_X(t)$ refers to the drift of X (in level) and σ_X to its volatility.

constant.[3] We adopt this terminal reward and will refer to ζ as the bequest-motive coefficient.

Kreps–Porteus SDU. The homogeneity conditions on f and A imply that we can write

$$f(c, v) = v\,F(c/v) \quad \text{and} \quad A(v) = -\gamma/v, \tag{2.4}$$

for some function F and some constant γ. It can be shown that γ is the coefficient of relative risk aversion for static wealth gambles.[4] We restrict our attention to $\gamma \geq 0$. The form of F determines the rate of time preference and the elasticity of intertemporal substitution. Let $\beta > 0$ denote the rate of time preference and let $\eta \geq 0$ denote the elasticity of intertemporal substitution. We will find the following reparametrization of η and γ useful:

$$\rho := 1 - 1/\eta \leq 1 \quad \text{and} \quad \alpha := 1 - \gamma \leq 1. \tag{2.5}$$

As we will see below (when we compute the interest rate from the utility gradient),

$$\beta = x\,F'(x) - F(x)\,\rho \quad \text{and} \quad \rho = 1 + \frac{x\,F''(x)}{F'(x)}. \tag{2.6}$$

Requiring β and ρ (or η) to be constant and $F(x)$ to be continuous in ρ produces[5]

$$F(x) = \beta\,u(x) \quad \text{where} \quad u(x) = \begin{cases} (x^\rho - 1)/\rho & \text{for } \rho \neq 0 \\ \log(x) & \text{for } \rho = 0. \end{cases} \tag{2.7}$$

As shown by Duffie and Epstein (1992a), these preferences allow a disentangling of attitudes toward risk from attitudes toward intertemporal substitution. The preference for early versus late resolution of uncertainty is characterized by the sign of $\alpha - \rho$. For $\alpha - \rho = 0$, Kreps–Porteus SDU specializes to time-separable preferences with power utility, for which the consumer is indifferent toward the timing of resolution of uncertainty. This locus is shown as the rectangular hyperbola $\eta\,\gamma = 1$ in Figure 1 (where γ is plotted on the vertical axis against η on the horizontal axis). For $\alpha - \rho < 0$ (or $\eta\,\gamma > 1$), the consumer prefers early resolution, while for $\alpha - \rho > 0$ (or $\eta\,\gamma < 1$), late resolution. We will discuss the other loci below.

To sum up, our homothetic SDU preferences are characterized by the vector of four parameters $(\beta, \eta, \gamma, \zeta)$, or, equivalently, $(\beta, \rho, \alpha, \zeta)$.

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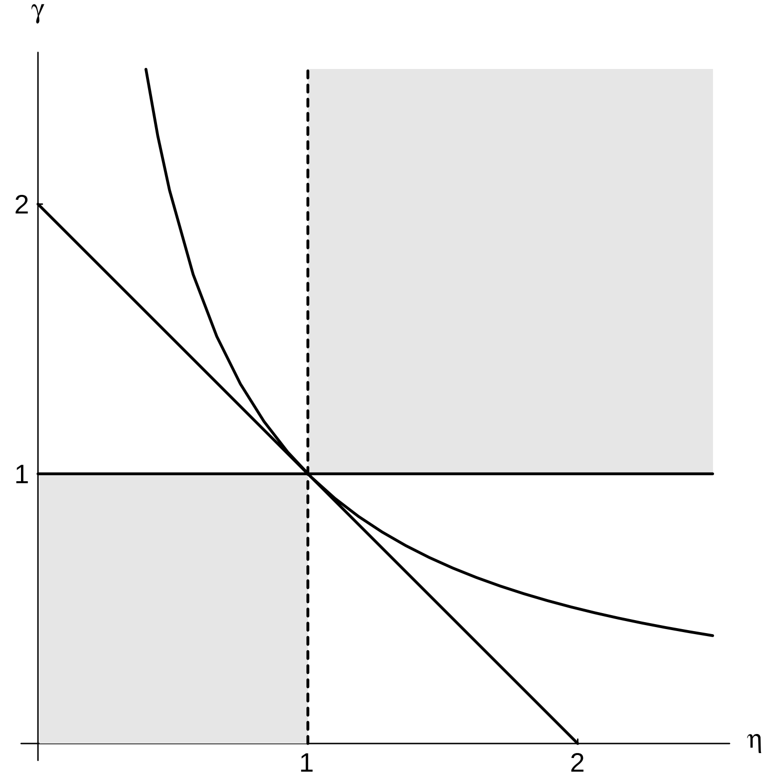
[3]The existing literature has by and largely adopted the terminal reward $\zeta = 0$. An exception is Schroder and Skiadas (1999) who provide a brief discussion of non-zero terminal rewards in general and an example using $\zeta = 1$ for $\gamma = 1$. As we show in Appendix C, a zero terminal reward implies that there is no solution when $\rho = 0$. A standard fix is to adopt the inhomogeneous terminal reward $G(T) = \xi > 0$, but in this case there is no single transformation $\Upsilon(\cdot)$ such that $\Upsilon(V(t))$ is linearly homogeneous in $\{c(s) \mid s \geq t\}$ simultaneously for all t.

[4]See Epstein (1992) for a discussion of the relation between the properties of a certainty equivalent in a dynamic setting and risk aversion in a static setting.

[5]This specification appears in Duffie and Epstein (1992a, p. 418). Schroder and Skiadas (1999) adopt a different parameterization: their γ is ρ, and their α is $(\alpha/\rho) - 1$ if $\rho \neq 0$ and otherwise their α is α.

FIGURE 1. The coefficient of relative risk aversion, γ, versus the elasticity of intertemporal substitution, η. The shaded areas show where $\alpha/\rho < 0$.



The central restriction for homogeneous SDU. The linear homogeneity of continuation utility in consumption implies $V(t) = c(t)\,\psi(t)$ for some process $\{\psi\}$; the terminal reward implies $\psi(T) = \zeta$. For strictly positive and finite ψ,[6] we can write

$$d\log(\psi(t)) = \widetilde{\mu}_\psi(t)\,dt + \sigma_\psi(t)^\top dW(t).$$

Ito's lemma applied to $V(t) = c(t)\,\psi(t)$ yields

$$\mu_V = V\left(\widetilde{\mu}_c + \widetilde{\mu}_\psi + \frac{1}{2}\,\|\sigma_c + \sigma_\psi\|^2\right) \quad \text{and} \quad \sigma_V = V\,(\sigma_c + \sigma_\psi).$$

Thus we can write (2.2) as

$$\widetilde{\mu}_c + \widetilde{\mu}_\psi + \alpha\,\frac{1}{2}\,\|\sigma_c + \sigma_\psi\|^2 + \beta\,u(1/\psi) = 0, \quad \text{subject to } \psi(T) = \zeta. \tag{2.8}$$

For a given process c, if the process ψ solves (2.8), then the process $c\,\psi$ is continuation utility, provided $\int_{s=0}^t \sigma_V(s)^\top dW(s)$ is a martingale.[7] Thus, whenever a solution

[6] As we discuss below, we can avoid technical problems at the boundary, where $\psi = 0$ or $\psi = \infty$ may occur, with a change of variables.

[7] See Proposition 3 in Schroder and Skiadas (1999).

to the underlying problem exists, the solution to (2.8) provides it. However, the solution to (2.8) does not provide the solution to the underlying problem unless the volatility of $c\,\psi$ is well behaved.

In general, (c, ψ) is not jointly Markovian. In a Markovian setting,[8] we have $\psi(t) = \Psi(X(t), t)$ for some function Ψ, where X is a vector of Markovian state variables. Given (2.8), the only way for the state variables to affect ψ is through $\widetilde{\mu}_c$ and σ_c. In effect, ψ summarizes all information about future opportunities contained in the dynamics of the *forcing variable* consumption. As such, we refer to ψ as the *information variable.*

The normalized aggregator and the utility gradient. Having characterized the utility of a given consumption process, we now turn to marginal utility, which provides the link to optimality. Duffie and Skiadas (1994) derive the Riesz representation of the utility gradient for a wide class of dynamic utilities including a normalized version of KP-SDU.[9]

As discussed by Duffie and Epstein (1992a), for any SDU there exists a normalized form $(\bar{f}, \bar{A})$ where $\bar{A} \equiv 0$. The normalization is achieved via an increasing monotonic transformation of V that produces a new continuation utility that is ordinally equivalent. Suppose we define $\bar{V}(t) := \Upsilon(V(t))$, where $\Upsilon(x)$ is twice-continuously differentiable and strictly increasing. Even though the change of variables has no effect on choices, it does change the form of the aggregator (through Itô's lemma). Let $\Phi(x)$ denote the inverse function of $\Upsilon(x)$ so that $\Phi(\Upsilon(x)) = x$. Then the aggregator for $\bar{V}$ is $(\bar{f}, \bar{A})$, where

$$\bar{f}(c, z) = f(c, \Phi(z))/\Phi'(z) \quad \text{and} \quad \bar{A}(z) = \Phi'(z)\, A(\Phi(z)) + \Phi''(z)/\Phi'(z),$$

and the terminal reward is $\bar{G}(T) = \Upsilon(G(T))$. If Υ is chosen to satisfy $\Upsilon''(x) - A(x)\,\Upsilon'(x) = 0$, then $\bar{A} = 0$. For homogeneous SDU, the normalizing transformation is $\Upsilon(x) = (x^\alpha - 1)/\alpha$, so that $\Phi(x) = (1 + \alpha\, x)^{1/\alpha}$, with the appropriate limits for $\alpha = 0$. For $\alpha \neq 0$ and $\rho \neq 0$, the normalized aggregator is:

$$\bar{f}(c, z) = (1 + \alpha\, z)\, \frac{\beta\left(\left(c\,(1 + \alpha\, z)^{-1/\alpha}\right)^\rho - 1\right)}{\rho}. \tag{2.9}$$

For the limiting cases, we have:

$$\bar{f}(c, z) = \begin{cases} \beta\,((c^\rho - 1)/\rho - z) & \text{for } \alpha = \rho \\ (\beta/\rho)\,(c^\rho\, e^{-\rho\, z} - 1) & \text{for } \alpha = 0 \\ (1 + \alpha\, z)\,\beta\,(\log(c) - (1/\alpha)\, \log(1 + \alpha\, z)) & \text{for } \rho = 0. \end{cases} \tag{2.10}$$

[8] See Section 5.

[9] See Duffie (1996) for a discussion of the utility gradient.

We can compute the partial derivatives of $\bar{f}(c, \bar{V})$ as defined in (2.9) and express them in terms of the un-normalized continuation utility:

$$\begin{aligned}\bar{f}_c(c, \Upsilon(V)) &= \beta\, u'(c/V)\, V^{\alpha-1} \\ &= \beta\, c^{\alpha-1}\, \psi^{\alpha-\rho}\end{aligned} \tag{2.11a}$$

$$\begin{aligned}\bar{f}_v(c, \Upsilon(V)) &= -\beta - (\rho-\alpha)\,\beta\, u(c/V) \\ &= -\beta - (\rho-\alpha)\,\beta\, u(1/\psi).\end{aligned} \tag{2.11b}$$

The utility gradient is expressed in terms of the *Gateaux derivative* of $\bar{V}(0)$. The Gateaux derivative $\bar{V}(t)$ at $\{c\}$ in the direction of $\{\hat{c}\}$ is defined by

$$\nabla\bar{V}(\{c\};\{\hat{c}\},t) = \lim_{\alpha\downarrow 0} \frac{\bar{V}^{\{c+\alpha\,\hat{c}\}}(t) - \bar{V}^{\{c\}}(t)}{\alpha}.$$

It can be expressed as

$$\nabla\bar{V}(\{c\};\{\hat{c}\},t) = E_t\left[\int_{s=t}^{T} \frac{\mathcal{G}(s)}{\mathcal{D}(t)}\,\hat{c}(s) + \frac{\mathcal{G}(T)}{\mathcal{D}(t)}\,\hat{c}(T)\right],$$

where

$$\mathcal{G}(t) = \mathcal{D}(t)\,\bar{f}_c(c(t), \bar{V}(t)), \qquad \text{for } t < T \tag{2.12a}$$

$$\mathcal{G}(T) = \mathcal{D}(T)\,\frac{d\bar{G}(T)}{dc(T)}, \tag{2.12b}$$

and

$$\mathcal{D}(t) := \exp\left\{\int_{s=0}^{t} \bar{f}_v(c(s), \bar{V}(s))\, ds\right\}.$$

$\mathcal{G}$ is the Riesz representation of this utility gradient. Since $\bar{V}$ is ordinally equivalent to V, the marginal rate of substitution for both $\bar{V}$ and V between time t and time s is given by $\mathcal{G}(s)/\mathcal{G}(t)$.

Supporting price system. We now use the Riesz representation of the utility gradient derive the price system (*i.e.*, the interest rate and price of risk) that supports the endowment. To support a consumption plan that we assume to be strictly positive (we restrict attention to interior solutions), prices must be aligned with marginal rates of substitution, which in the present context means that the state-price deflator must be colinear with $\mathcal{G}(t)$, so that $m(t) = a\,\mathcal{G}(t)$ for some scalar $a > 0$.[10] We assume in this paper that this first-order condition is also sufficient for the optimality of the solution.[11]

[10] See Appendix A for a discussion of the state–price deflator.
[11] See Schroder and Skiadas (1999) for some results.

In accord with (A.1), applying Itô's lemma to $a\,\mathcal{G}(t)$, where $\mathcal{G}(t)$ is given by (2.12), delivers the short rate r and the price of risk λ:[12]

$$r = \beta + (1-\rho)\,\widetilde{\mu}_c - \alpha\,(\rho-\alpha)\,\frac{1}{2}\,\|\sigma_c + \sigma_\psi\|^2 - \frac{1}{2}\,\|\lambda\|^2 \tag{2.13a}$$

$$\lambda = (1-\alpha)\,\sigma_c + (\rho-\alpha)\,\sigma_\psi, \tag{2.13b}$$

where we have used (2.8) to eliminate $\widetilde{\mu}_\psi + \beta\,u(1/\psi)$ from r. Note that σ_ψ enters the price of risk with a sign that depends on whether early or late resolution of uncertainty is preferred. We see that $\rho = \alpha$ delivers the well-known expressions for r and λ under the C–CAPM.[13]

A representation with explicit discounting. Define $\hat{V}(t) = u(V(t))$. In this case $\Phi(x) = u^{-1}(x) = (1+\rho\,x)^{1/\rho}$. The aggregator for $\hat{V}$ is

$$\hat{f}(c,z) = \beta\,(u(c) - z) \quad \text{and} \quad \hat{A}(z) = \frac{\alpha-\rho}{1+\rho\,z}.$$

where $1+\rho\,z$ is positive. It then follows from (2.1) that[14]

$$\hat{V}(t) = E_t\left[\int_{s=t}^{T} \beta\,e^{-\beta\,(s-t)}\left\{u(c(s)) + \frac{1}{\beta}\left(\frac{\alpha-\rho}{1+\rho\,\hat{V}(s)}\right)\frac{1}{2}\,\|\sigma_{\hat{V}}(s)\|^2\right\}ds + e^{-\beta\,(T-t)}\,u(G(T))\right]. \tag{2.14}$$

Recall that with standard preferences $\alpha - \rho = 0$, and that $\alpha - \rho < 0$ is associated with preference for early resolution of uncertainty with respect to utility, in which case uncertainty about continuation utility reduces current utility.

Using $\hat{V}(t) = u(c(t)\,\psi(t))$, $\|\sigma_{\hat{V}}(s)\|^2/(1+\rho\,\hat{V}(s)) = V(s)^\rho\,\|\sigma_V(s)/V(s)\|^2$, and $u(x)/y^\rho + u(1/y) = u(x/y)$, we can rewrite (2.14) as

$$u(\psi(t)) = E_t\left[\int_{s=t}^{T} \beta\,e^{-\beta\,(s-t)}\,u\left(\frac{c(s)}{c(t)}\right)ds + e^{-\beta\,(T-t)}\,u\left(\zeta\,\frac{c(T)}{c(t)}\right)\right] + (\alpha-\rho)\,E_t\left[\int_{s=t}^{T} e^{-\beta\,(s-t)}\,\psi(s)^\rho\,\frac{1}{2}\,\|\sigma_c(s)+\sigma_\psi(s)\|^2\,ds\right]. \tag{2.15}$$

This provides a semi-explicit representation for the information variable; for standard preferences ($\alpha - \rho = 0$), the representation is fully explicit. We can obtain a useful approximation from (2.15) near $\rho = 0$ as follows: multiply both sides of

[12]Note that if consumption grows as a constant rate (so that $\sigma_c = 0$ and $\sigma_V = 0$), then $r = \beta + (1-\rho)\,\widetilde{\mu}_c$. When the growth rate is zero (constant consumption), the interest rate equals β, the rate of time preference. Moreover, since $\widetilde{\mu}_c = (1-\rho)^{-1}\,(r-\beta)$, we see that $d\widetilde{\mu}_c/dr = (1-\rho)^{-1} = \eta$, the elasticity of intertemporal substitution.

[13]This is consistent with Theorem 2(a) (under condition I) in Schroder and Skiadas (1999).

[14]Related representations appear in Duffie and Lions (1992) and Schroder and Skiadas (1999).

(2.15) by ρ, add 1, and evaluate at $\rho = 0$ to produce

$$\log(\psi(t)) = E_t\left[\int_{s=t}^{T} \beta\, e^{-\beta\,(s-t)} \log\left(\frac{c(s)}{c(t)}\right) ds + e^{-\beta\,(T-t)} \log\left(\zeta\,\frac{c(T)}{c(t)}\right)\right] + \alpha\, E_t\left[\int_{s=t}^{T} e^{-\beta\,(s-t)}\, \frac{1}{2}\, \|\sigma_c(s) + \sigma_\psi(s)\|^2\, ds\right] + \mathcal{O}(\rho). \quad (2.16)$$

Risk aversion and risk neutrality. In this section, we emphasize that setting the coefficient of risk aversion $\gamma = 0$ is not sufficient to guarantee risk neutrality. Duffie and Epstein (1992b) define risk aversion for SDU as follows. Let $\{c\}$ denote an arbitrary consumption process taken from some appropriate space. Define a new consumption process $\{\bar{c}\}$ where $\bar{c}(t) = E_0[c(t)]$. A utility function $U(\{c\}) = V^{\{c\}}(0)$ is *risk averse* if $U(\{c\}) \leq U(\{\bar{c}\})$. We define *risk neutrality* in a parallel fashion: A utility function is risk-neutral if $U(\{c\}) = U(\{\bar{c}\})$. For $(\zeta, \rho) \neq (0, 0)$, $V^{\{c\}}(0) = c(0)\, \psi^{\{c\}}(0)$. Therefore it is enough to compare $\psi^{\{c\}}(0)$ with $\psi^{\{\bar{c}\}}(0)$. We can use (2.15) to express $\left(\psi^{\{c\}}(0)\right)^\rho$ and $\left(\psi^{\{\bar{c}\}}(0)\right)^\rho$.[15] Multiply both sides of (2.15) by ρ and add one:

$$\left(\psi^{\{c\}}(0)\right)^\rho = \int_{s=0}^{T} \beta\, e^{-\beta\, s}\, E_0\left[\left(\frac{c(s)}{c(0)}\right)^\rho\right] ds + e^{-\beta\, T}\, E_0\left[\left(\zeta\,\frac{c(T)}{c(0)}\right)^\rho\right] + \rho(\alpha - \rho)\, E_0\left[\int_{s=0}^{T} e^{-\beta\, s}\, \psi(s)^\rho\, \frac{1}{2}\, \|\sigma_c(s) + \sigma_\psi(s)\|^2\, ds\right] \quad (2.17)$$

and

$$\left(\psi^{\{\bar{c}\}}(0)\right)^\rho = \int_{s=0}^{T} \beta\, e^{-\beta\, s} \left(\frac{E_0\left[c(s)\right]}{c(0)}\right)^\rho ds + e^{-\beta\, T} \left(\zeta\,\frac{E_0\left[c(T)\right]}{c(0)}\right)^\rho. \quad (2.18)$$

For $\rho = 1$,

$$\psi^{\{\bar{c}\}}(0) - \psi^{\{c\}}(0) = \gamma\, E_0\left[\int_{s=0}^{T} e^{-\beta\, s}\, \psi(s)\, \frac{1}{2}\, \|\sigma_c(s) + \sigma_\psi(s)\|^2\, ds\right] \geq 0,$$

with equality if $\gamma = 0$ (or if there is no state variation). For $\alpha = \rho$,

$$U(\{\bar{c}\})^\rho - U(\{c\})^\rho = \int_{s=0}^{T} \beta\, e^{-\beta\, s}\, \left(E_0\left[c(s)\right]^\rho - E_0\left[c(s)^\rho\right]\right) ds + \zeta^\rho\, e^{-\beta\, T}\, \left(E_0\left[c(T)\right]^\rho - E_0\left[c(T)^\rho\right]\right). \quad (2.19)$$

Regardless of the sign of ρ, (2.19) implies $U(\{\bar{c}\}) - U(\{c\}) \geq 0$, with equality if $\rho = 1$ (or if $\{c\}$ is deterministic). Evidently, risk-neutrality requires $\alpha = \rho = 1$ (*i.e.*, $\gamma = 0$ and $\eta = \infty$). Of course, these are the conditions that ensure the price of risk is zero. In addition these conditions ensure that the interest rate is constant, a point made by Cox, Ingersoll, Jr., and Ross (1981) in a related setting.

[15] For $\rho = 0$, we can use (2.16).

3. Optimal consumption and portfolio

In this section we derive the supporting returns process and then solve for optimal consumption and the optimal portfolio.

Wealth and consumption. Let k denote wealth. Wealth is the present value of the consumption endowment (there is no harm in identifying the state-price deflator with the utility gradient, since they are colinear):

$$k(t) = Z(t)/\mathcal{G}(t), \tag{3.1}$$

where

$$Z(t) := E_t\left[\int_{s=t}^{T} \mathcal{G}(s)\,c(s)\,ds + \mathcal{G}(T)\,c(T)\right], \tag{3.2}$$

and where $dZ = -(\mathcal{G}\,c)\,dt + \bar{\sigma}_Z^\top dW$, for some $\bar{\sigma}_Z$. Applying Itô's lemma to $k = Z/\mathcal{G}$ produces

$$dk = k\,\frac{d\phi}{\phi} - c\,dt, \tag{3.3}$$

where $d\phi/\phi = \mu_\phi\,dt + \sigma_\phi^\top dW$ and

$$\mu_\phi = r + \lambda^\top \sigma_\phi \qquad \text{and} \qquad \sigma_\phi = \lambda + \bar{\sigma}_Z/Z. \tag{3.4}$$

Thus $d\phi/\phi$ is the stochastic rate of return that supports the endowment. The results in this section so far do not depend on any special assumptions about preferences. To get more specific results (and in particular to identify $\bar{\sigma}_Z/Z$), we now assume homogeneous SDU.

The wealth–consumption ratio. We now establish the relation between the wealth–consumption ratio and the information variable. Rearranging the right-hand side of (3.1), we can express wealth as

$$k(t) = \frac{\left(\frac{Z(t)}{\mathcal{D}(t)\,c(t)}\right)}{\left(\frac{\mathcal{G}(t)}{\mathcal{D}(t)}\right)}\,c(t), \tag{3.5}$$

where $\mathcal{G}(t)/\mathcal{D}(t) = \bar{f}_c(c(t), \bar{V}(t)) = \beta\,c(t)^{\alpha-1}\,\psi(t)^{\alpha-\rho}$ and

$$\frac{Z(t)}{\mathcal{D}(t)\,c(t)} = \frac{\nabla\bar{V}(\{c\},\{c\},t)}{c(t)}. \tag{3.6}$$

The right-hand side of (3.6) is the Gateaux derivative of $\bar{V}(t)$ evaluated at the endowment, in the direction of the endowment, per unit of current consumption. It measures the marginal (continuation) utility of a permanent, proportional increase in consumption:

$$\frac{\nabla\bar{V}(\{c\},\{c\},t)}{c(t)} = \frac{\partial\bar{V}(t)}{\partial c(t)} = \frac{\partial}{\partial c(t)}\left(\frac{\big(c(t)\,\psi(t)\big)^\alpha - 1}{\alpha}\right) = c(t)^{\alpha-1}\,\psi(t)^\alpha. \tag{3.7}$$

Thus we can write (3.5) as

$$k(t) = \left(\frac{\psi(t)^\rho}{\beta}\right) c(t).$$

Denoting the wealth–consumption ratio $\pi := k/c$, we have $\pi = \psi^\rho/\beta$.[16] Given the homotheticity of utility and the linearity of investment technology implicit in the conceptual experiment, the wealth–consumption ratio is scale-free, depending only on the information variable.

Before returning to the supporting returns process, we note some implications of the relation between the wealth–consumption ratio and the information variable. The boundary condition $\psi(T) = \zeta$ implies $\pi(T) = \zeta^\rho/\beta$. For $\rho < 0$, $\pi(T)$ and ζ are inversely related; in particular, for $\pi(T) = 0$ we must have $\zeta = \infty$.[17] For $\rho = 0$, $\pi(t) = 1/\beta$ for all $t \leq T$. With $\zeta = 1$, the terminal wealth–consumption ratio is $1/\beta$ for all values of ρ. The first-order approximation for π around $\rho = 0$ can be obtained from (2.16) using $\pi = (1+\rho\,\log(\psi))/\beta + \mathcal{O}(\rho^2)$.[18] We can transform (2.15) into an expression for the wealth–consumption ratio by multiplying both sides by ρ, adding 1, and dividing by β:

$$\pi(t) = E_t\left[\int_{s=t}^{T} e^{-\beta\,(s-t)}\left(\frac{c(s)}{c(t)}\right)^\rho ds + e^{-\beta\,(T-t)}\left(\frac{c(T)}{c(t)}\right)^\rho \frac{\zeta^\rho}{\beta}\right] + (\alpha/\rho - 1)\,E_t\left[\int_{s=t}^{T} e^{-\beta\,(s-t)}\,\pi(s)\,\frac{1}{2}\,\|\rho\,\sigma_c(s) + \sigma_\pi(s)\|^2\,ds\right]. \quad (3.8)$$

The wealth–consumption ratio is composed of two terms. The first term is the wealth–consumption ratio for the C–CAPM, while the second term, which involves the volatility of consumption, is present only when preferences are not additive. For $\rho = 0$, the first term is $1/\beta$ and the second term is zero.

Supporting returns process. In deriving the relation between the wealth–consumption ratio and the information variable, we have identified $\bar{\sigma}_Z/Z$: Equations (3.6) and (3.7) imply $Z(t) = \mathcal{D}(t)\,\big(c(t)\,\psi(t)\big)^\alpha$, so that $\bar{\sigma}_Z/Z = \alpha\,(\sigma_c + \sigma_\psi)$. We can write (3.4) as

$$\widetilde{\mu}_\phi = r + \lambda^\top\sigma_\phi - \frac{1}{2}\,\|\sigma_\phi\|^2 \quad (3.9a)$$

$$\sigma_\phi = \lambda + \alpha\,(\sigma_c + \sigma_\psi). \quad (3.9b)$$

Note that the returns process is independent of wealth. Together (2.13) and (3.9) establish the relations among the dynamics of consumption, the supporting price system, and the supporting returns process. For example, we can use (2.13b) to

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[16] We can compute this directly in terms of the homogeneous representation of utility as $(\partial V/\partial c)/f_c = \psi^\rho/\beta$.

[17] Of course, for $\rho \neq 0$, we could reparameterize the bequest-motive parameter: $\hat{\zeta} := \zeta^\rho$. Then $\pi(T) = \hat{\zeta}/\beta$ for all $\rho \neq 0$.

[18] The approximations in Campbell (1993) are implicitly based on this approximation.

eliminate σ_c from (3.9b) and express the price of risk as

$$\lambda = \gamma\,\sigma_\phi + (1-\gamma)\left(\frac{\sigma_\psi}{\eta}\right). \tag{3.10}$$

Evidently risk premia depend on the covariance with the returns process and with the information variable. Campbell (1993) derives an equivalent expression for risk premia (his Equation (25)), which he refers to as the "cross-sectional asset pricing formula that makes no reference to consumption." We can obtain Campbell's parameterization with a change of variables. Define $\omega := \psi^{1/\eta}$. Then $\sigma_\psi = \eta\,\sigma_\omega$, and (3.10) becomes $\lambda = \gamma\,\sigma_\phi + (1-\gamma)\,\sigma_\omega$.

Optimal consumption. Up to this point, we have treated the consumption process as given: Current opportunities have been given by current consumption and future opportunities have been determined by the dynamics of consumption. Now we change perspective. Consumption is no longer given exogenously. Instead, current opportunities are given by current wealth and future opportunities are determined by stochastic investment returns—either directly via the investment technology or indirectly via the price system.

The dynamics of wealth as given in (3.3) embody the consumption–investment trade-off. We can interpret $d\phi/\phi$ as the return on optimally invested wealth and σ_ϕ as the volatility of the optimal portfolio. We refer to ϕ as the *capital account*, which grows at the rate on a marginal investment. The source of returns could be a portfolio of securities, or it could be a single stochastic investment technology. In either case, ϕ tracks the outcome of the following investment strategy: invest one unit of the consumption good in the returns process at time zero and thereafter continuously reinvest the proceeds.

In the current setting, the information variable will summarize all relevant information about future opportunities as reflected in the dynamics of either the state–price deflator or the capital account. In other words, the information variable must conform to the dynamics of the *forcing variable*. Previously, in the endowment setting, the forcing variable was consumption. We now allow the forcing variable to be the state–price deflator or the capital account. The restriction ψ must satisfy when the forcing variable is the state–price deflator is obtained by eliminating $\widetilde{\mu}_c$ and σ_c from (2.8) using (2.13). Similarly, the restriction ψ must satisfy when the forcing variable is the capital account is obtained by eliminating $\widetilde{\mu}_c$ and σ_c from (2.8) using (2.13) and (3.9).

Thus there are three versions of (2.8), the central restriction on the information variable, each depending on a different choice for the forcing variable. It is convenient to formally unify all three restrictions. To that end, we denote the generic forcing variable y and its dynamics $d\log(y(t)) = \widetilde{\mu}_y(t)\,dt + \sigma_y(t)^\top dW(t)$, where y is either consumption (c), the capital account (ϕ), or the inverse of the state–price deflator ($1/m$). We can write all three restrictions as

$$a_0 + a_1\,\widetilde{\mu}_y + \widetilde{\mu}_\psi + a_2\,\frac{1}{2}\,\|\sigma_\psi + a_1\,\sigma_y\|^2 + \beta\,u(1/\psi) = 0, \quad \text{subject to } \psi(T) = \zeta, \tag{3.11}$$

Table 1. Coefficients for Equation (3.11).

y	a_0	a_1	a_2
c	0	1	$1-\gamma$
ϕ	$-\beta\,\eta$	η	$(1-\gamma)/\eta$
$1/m$	$-\beta\,\eta$	η	$(1-\gamma)/(\eta\,\gamma)$

where the coefficients a_i are given in Table 1. Given the dynamics of the forcing variable and the solution to (3.11), we can use (2.13) and (3.9) to compute the dynamics of the remaining variables. We can turn (3.11) into a PDE by adopting the Markovian structure described in Section 5.

Optimal portfolio. As we suggested above, the capital account can be thought of as the optimal portfolio of securities. Here we solve for the portfolio weights.

The investment opportunity set can be characterized by n risky securities with dynamics of the form

$$\frac{d\phi_i}{\phi_i} = \mu_{\phi_i}\,dt + \sigma_{\phi_i}^\top dW, \tag{3.12}$$

where W is composed of ℓ orthogonal Brownian motions. The expected return on security i obeys the absence-of-arbitrage condition

$$\mu_{\phi_i} = r + \sigma_{\phi_i}^\top \lambda. \tag{3.13}$$

(The dynamics of the risky assets reflect the reinvestment of any dividends paid.) Define $M_\phi := (\mu_{\phi_1}, \ldots, \mu_{\phi_n})^\top$, let Σ_ϕ denote the $\ell \times n$ matrix whose i-th column is σ_{ϕ_i}, and let $\underline{r}$ denote a vector of length n with each element equal to r. Then we can stack together the n equations (3.13) as

$$M_\phi = \underline{r} + \Sigma_\phi^\top \lambda. \tag{3.14}$$

In addition there is a money-market account (MMA), the value of which is b, where $db/b = r\,dt$.

A portfolio can be characterized by a vector of weights, $w(t) := (w_1(t), \ldots, w_n(t))^\top$, for the risky securities and a weight $w_0(t)$ for the MMA, such that $\sum_{i=0}^n w_i = 1$. The value of a portfolio evolves as follows:

$$\frac{d\phi}{\phi} = w_0\,\frac{db}{b} + \sum_{i=1}^n w_i\,\frac{d\phi_i}{\phi_i} = \mu_\phi\,dt + \sigma_\phi^\top dW,$$

where $\mu_\phi = w_0\,r + M_\phi^\top w$ and

$$\sigma_\phi = \Sigma_\phi\,w. \tag{3.15}$$

Eliminating σ_ϕ from (3.10) and (3.15) produces

$$\Sigma_\phi\, w = \left(\frac{1}{\gamma}\right) \lambda + \left(\frac{1-\gamma}{\eta\,\gamma}\right) \sigma_\psi. \tag{3.16}$$

Together, (3.14) and (3.16) comprise a linear system of $n+\ell$ equations in the $n+\ell$ unknowns w and λ. Assuming Σ_ϕ has full rank, the solution to this system is

$$\lambda = \Sigma_\phi \left(\Sigma_\phi^\top \Sigma_\phi\right)^{-1} (M_\phi - \underline{r}) + \left(\frac{1-\gamma}{\eta}\right) \left(\Sigma_\phi \left(\Sigma_\phi^\top \Sigma_\phi\right)^{-1} \Sigma_\phi^\top - I_\ell\right) \sigma_\psi \tag{3.17a}$$

$$w = \left(\frac{1}{\gamma}\right) \left(\Sigma_\phi^\top \Sigma_\phi\right)^{-1} (M_\phi - \underline{r}) + \left(\frac{1-\gamma}{\eta\,\gamma}\right) \left(\Sigma_\phi^\top \Sigma_\phi\right)^{-1} \Sigma_\phi^\top \sigma_\psi, \tag{3.17b}$$

where I_ℓ denote the $\ell \times \ell$ identity matrix. We see that the portfolio weights are composed of two terms. The first term is the so-called "myopic" component of portfolio demand, while the second term constitutes a hedge against changes in investment opportunities. As the horizon approaches zero, the boundary condition requires σ_ψ to go to zero as the need to hedge against changes in future opportunities attenuates.

In the complete markets setting $n = \ell$, Σ_ϕ is invertible, and (3.17) specializes to

$$\lambda = \left(\Sigma_\phi^\top\right)^{-1} (M_\phi - \underline{r}) \tag{3.18a}$$

$$w = \left(\frac{1}{\gamma}\right) \Sigma_\phi^{-1} \left(\Sigma_\phi^\top\right)^{-1} (M_\phi - \underline{r}) + \left(\frac{1-\gamma}{\eta\,\gamma}\right) \Sigma_\phi^{-1} \sigma_\psi. \tag{3.18b}$$

In this case, the solution for λ is independent of σ_ψ, and therefore we can treat $1/m$ as the forcing variable and solve (3.11) for ψ. Given the solution for ψ, (3.18b) delivers the solution for w. By contrast, suppose there are $n < \ell$ securities. In this case, λ involves σ_ψ, so that $1/m$ cannot be specified exogenously. Nevertheless, we can insert the expression for λ in (3.17a) into (3.11) for $y = 1/m$, thereby extending the incomplete-markets results of He and Pearson (1991) to recursive preferences.

The general equilibrium production problem. In a representative-agent general equilibrium model, we interpret $k(t)$ as the value of the capital stock and $d\phi/\phi$ as the return on the aggregate investment portfolio—*i.e.*, the return on the market portfolio. For the purpose of studying general equilibrium, we can reinterpret the securities as linear production technologies subject to random shocks as in Cox, Ingersoll, Jr., and Ross (1985a).[19] In a general equilibrium, the interest rate is endogenous and there is no borrowing or lending. Thus we require $w_0 = 0$ or equivalently $\sum_{i=1}^n w_i = 1$. To solve for the equilibrium, add this equation to the system (3.16) and solve (3.14) and (3.16) for r, λ, and w, in terms of M_ϕ and Σ_ϕ. These expressions for r and λ can be inserted into (3.11), and that equation can then be solved for ψ. By itself, the capital account is a special case where $n = 1$ and $w_1 = 1$.

[19]Some activities may slip in and out of the optimal portfolio. At the expense of some notation for keeping track of which activities are in the optimal portfolio at time t, we do not need to assume that the set of activities in the portfolio never changes.

At this level of analysis, we ignore the portfolio allocation problem, except to require zero net investment in the money-market account, treating the an economy as one with a single investment opportunity.[20]

4. The dividend-denominated term-structure

In this section we demonstrate that the wealth–consumption ratio is the value of a fixed-income asset, and we use this perspective to analyze various aspects of the model and its solution.

Asset prices and the wealth–consumption ratio. Using the Riesz representation of the utility gradient as state-price deflator, we can price assets. It is convenient to reparameterize the utility gradient for time T. Define

$$m(t) := \mathcal{D}(t)\,\bar{f}_c(c(t), \bar{V}(t)), \quad \text{for } t \le T,$$

so that $\mathcal{G}(t) = m(t)$ for $t < T$ and, in view of

$$\frac{d\bar{G}(T)}{dc(T)} = \xi^\alpha\, c(T)^{\alpha-1} = \bar{f}_c(c(T), \Upsilon(c(T)\,\xi))\,\frac{\xi^\rho}{\beta},$$

$\mathcal{G}(T) = m(T)\,\xi^\rho/\beta$. Note that $m(t)$ is not really the state-price deflator in this economy, since there is a wedge between $m(T)$ and $\mathcal{G}(T)$. But $m(t)$ is more convenient to use for what follows, and it can be treated as the state-price deflator by adjusting terminal payoffs by the factor ζ^ρ/β. For example, the consumer's wealth is

$$k(t) = E_t\left[\int_{s=t}^T \left(\frac{m(s)}{m(t)}\right) c(s)\,ds + \left(\frac{m(T)}{m(t)}\right)\frac{\zeta^\rho}{\beta}\,c(T)\right]. \tag{4.1}$$

In (4.1), we may think of the consumption endowment as the dividend that accrues to the asset "wealth." The final "lump-sum payment," $(\zeta^\rho/\beta)\,c(T)$, reflects the terminal reward. Dividing both sides of (4.1) by $c(t)$ produces

$$\pi(t) = E_t\left[\int_{s=t}^T \frac{m_d(s)}{m_d(t)}\,ds + \frac{m_d(T)}{m_d(t)}\,\frac{\zeta^\rho}{\beta}\right] = \int_{s=t}^T p_d(t,s)\,ds + \frac{\zeta^\rho}{\beta}\,p_d(t,T), \tag{4.2}$$

where $m_d(t) = m(t)\,c(t)$ and $p_d(t,s) = E_t[m_d(s)/m_d(t)]$ is the value at time t of a zero-coupon bond that pays one unit of the dividend (in this case, the consumption endowment) at time s. Equation (4.2) shows that $\pi(t)$ is the value of a *dividend-denominated* coupon-bond with face value ζ^ρ/β and coupon rate β/ζ^ρ. (If $\zeta^\rho = 0$, then $\pi(t)$ is the value of an annuity.) Formally, $m_d(t)$ is the state–price deflator where the numeraire has been changed from units of the consumption good to units of the endowment process (with terminal payoffs also adjusted by factor ζ^ρ/β).[21]

By applying Itô's lemma to $m_d = m\,c$, we can compute the dividend-denominated interest rate and price of risk in terms of the real interest rate and price of risk and

[20] This is the setting that Campbell (1993) studies. Campbell's "news about future returns on invested wealth" is the volatility of ψ^η when the forcing variable is the capital account.

[21] See Appendix A for a discussion of changing numeraires.

the dynamics of consumption:

$$r_d = r - \left(\widetilde{\mu}_c + \frac{1}{2}\|\sigma_c\|^2\right) + \lambda^\top \sigma_c, \tag{4.3a}$$

$$\lambda_d = \lambda - \sigma_c. \tag{4.3b}$$

Using (2.13) we can eliminate the real interest rate and price of risk, expressing the dividend-denominated interest rate and price of risk in terms of the dynamics of consumption and the volatility of the information variable:

$$r_d = \beta - \rho\left\{\widetilde{\mu}_c + \alpha\,\frac{1}{2}\|\sigma_c\|^2 + (\rho-\alpha)\frac{1}{2}\|\sigma_\psi\|^2\right\} \tag{4.4a}$$

$$\lambda_d = -\alpha\,\sigma_c + (\rho-\alpha)\,\sigma_\psi. \tag{4.4b}$$

Because π is the value of an asset (when measured in units of the endowment process), the drift of π will be determined by the martingale property of deflated gains:

$$\bar{\mu}_\pi + 1 = r_d\,\pi + \lambda_d^\top \bar{\sigma}_\pi, \quad \text{subject to } \pi(T) = \zeta^\rho/\beta, \tag{4.5}$$

where the (absolute) dynamics of π are given by $d\pi = \bar{\mu}_\pi\,dt + \bar{\sigma}_\pi^\top dW$. For $\rho \neq 0$, we can use the change of variables $\psi = (\beta\,\pi)^{1/\rho}$ to show that (4.5) is simply a restatement of (2.8).[22]

For $\rho = 0$, the dividend-denominated interest rate is constant, $r_d = \beta$, and $\pi(T) = 1/\beta$. In this case, the solution to (4.5) is $\pi(t) = 1/\beta$ for all $t \le T$. Even so, we can use (4.5) to solve for ψ as follows. Since the solution to π in (4.5) depends on ρ in a continuous way, we can evaluate the following:

$$\log(\psi)|_{\rho=0} = \lim_{\rho\to 0}\,\log\left((\beta\,\pi)^{1/\rho}\right) = \left.\frac{d\log(\pi)}{d\rho}\right|_{\rho=0} = \beta\,\left.\frac{d\pi}{d\rho}\right|_{\rho=0}. \tag{4.6}$$

The wealth–consumption ratio and the three forcing processes. Given the supporting price system and the supporting returns process, we can express the dividend-denominated interest rate and price of risk in terms of any of the three forcing variables as

$$r_d = d_0 + d_1\left(\widetilde{\mu}_y + d_2\,\frac{1}{2}\|\sigma_y\|^2\right) - (\varepsilon/d_1)\frac{1}{2}\left\|\frac{\bar{\sigma}_\pi}{\pi}\right\|^2 \tag{4.7a}$$

$$\lambda_d = -d_2\,\sigma_y + (\varepsilon/d_1)\,\frac{\bar{\sigma}_\pi}{\pi}, \tag{4.7b}$$

[22]Note that even for $\zeta^\rho = 0$, $\lim_{t\to T}\ \bar{\sigma}_\pi(t)/\pi(t) = 0$, since

$$\bar{\sigma}_\pi(t) = \int_{s=t}^T p_d(t,s)\,\sigma_{p_d}(t,s)\,ds \quad\text{and}\quad \pi(t) = \int_{s=t}^T p_d(t,s)\,ds,$$

where $\sigma_{p_d}(T,T) = 0$ and $p_d(T,T) = 1$. By contrast, $\lim_{t\to T}\ \bar{\mu}_\pi(t)/\pi(t) = \infty$. We can express the dynamics of $\log(\psi)$ in terms of the dynamics of π:

$$\widetilde{\mu}_\psi(t) = \frac{1}{\rho}\left(\frac{\bar{\mu}_\pi(t)}{\pi(t)} - \frac{1}{2}\left\|\frac{\bar{\sigma}_\pi(t)}{\pi(t)}\right\|^2\right) \quad\text{and}\quad \sigma_\psi(t) = \frac{1}{\rho}\left(\frac{\bar{\sigma}_\pi(t)}{\pi(t)}\right).$$

where the coefficients d_i are given in Table 2. In effect we have reexpressed (3.11) as (4.5) for $\rho \neq 0$ by the change of variable from ψ to π. When either $\varepsilon = 0$ or $\bar{\sigma}_\pi = 0$, r_d and λ_d are exogenously determined by the dynamics of the forcing process. In this case we can compute the wealth–consumption ratio from (4.2) by solving

$$\mu_{p_d}(t,T) = r_d(t) + \lambda_d(t)^\top \sigma_{p_d}(t,T), \qquad \text{subject to } p_d(T,T) = 1,$$

for $p_d(t,T)$.

TABLE 2. The coefficients of Equation (4.7) in terms of the preference parameters (columns 2–5), and the feasibility of limiting parameter values (columns 6–9).

y	d_0	d_1	d_2	$\varepsilon = d_1 + d_2$	$\eta = 0$	$\eta = \infty$	$\gamma = 0$	$\gamma = \infty$
c	β	$1/\eta - 1$	$1 - \gamma$	$1/\eta - \gamma$	no	yes	yes	no
ϕ	$\eta\,\beta$	$1 - \eta$	$1 - \gamma$	$2 - \eta - \gamma$	yes	no	yes	no
$1/m$	$\eta\,\beta$	$1 - \eta$	$1/\gamma - 1$	$1/\gamma - \eta$	yes	no	no	yes

Consider the case where the dividend-denominated interest rate is deterministic. When r_d is deterministic, the wealth–consumption ratio is given by (4.2) where $p_d(t,s) = \exp\bigl(-\int_{u=t}^{s} r_d(u)\,du\bigr)$. Hence π is deterministic, which implies $\bar{\sigma}_\pi = 0$ and thus $r_d = d_0 + d_1\,\bigl(\widetilde{\mu}_y + d_2\,\frac{1}{2}\,\|\sigma_y\|^2\bigr)$. In this case either (i) $d_1 = 0$ (*i.e.*, $\rho = 0$) or (ii) $\widetilde{\mu}_y + d_2\,\frac{1}{2}\,\|\sigma_y\|^2$ is deterministic. For constant r_d, the solution specializes to

$$\pi(t) = \frac{1 - e^{-r_d\,(T-t)}}{r_d} + \frac{\zeta^\rho\, e^{-r_d\,(T-t)}}{\beta}. \tag{4.8}$$

Utility gradient in terms of observables. In this section, we show where the condition $\varepsilon = 0$ comes from.

Having established the relation between the capital account, the consumption process, and the wealth–consumption ratio, we can eliminate the unobservable ψ from the utility gradient, following Epstein and Zin (1991), as long as $\rho \neq 0$. For $\rho \neq 0$, we can write the utility gradient as

$$\mathcal{G}(t) = \beta\, e^{-\beta\,(\alpha/\rho)\,t} \left\{ \exp\left(\int_{s=0}^{t} (\beta/\rho)\,\psi(s)^{-\rho}\,ds \right) \psi(t) \right\}^{\alpha-\rho} c(t)^{\alpha-1}. \tag{4.9}$$

Using $k = c\,\psi^\rho/\beta$, we can reexpress (3.3) as

$$d\log(\phi) = d\log(c) + \rho\, d\log(\psi) + \beta\,\psi^{-\rho}\,dt. \tag{4.10}$$

Integrating both sides of (4.10) and rearranging produces

$$C_0\,\frac{c(t)}{\phi(t)} = \left\{ \exp\left(\int_{s=0}^{t} (\beta/\rho)\,\psi(s)^{-\rho}\,ds \right) \psi(t) \right\}^{-\rho},$$

where $C_0 = \phi(0)\,\psi(0)^{-\rho}/c(0)$. We can use this relation to eliminate the expression in braces from (4.9):

$$\mathcal{G}(t) = e^{-\beta\,(\alpha/\rho)\,t}\,c(t)^{-(\alpha/\rho)\,(1-\rho)}\,\phi(t)^{(\alpha/\rho)-1}, \tag{4.11}$$

where we have suppressed the constant of proportionality. Identifying the utility gradient with the state–price deflator and applying Itô's lemma yields a convenient representation for the short rate r and the price of risk λ in terms of the observable dynamics of the growth rate of consumption and the return on the market portfolio ϕ:

$$r(t) = \beta\,(\alpha/\rho) + (\alpha/\rho)\,(1-\rho)\,\widetilde{\mu}_c(t) + (1-\alpha/\rho)\,\widetilde{\mu}_\phi(t) - \frac{1}{2}\,\|\lambda(t)\|^2 \tag{4.12a}$$

$$\lambda(t) = (\alpha/\rho)\,(1-\rho)\,\sigma_c(t) + (1-\alpha/\rho)\,\sigma_\phi(t). \tag{4.12b}$$

Note that $\alpha/\rho = 1$ (*i.e.*, $\eta\,\gamma = 1$) delivers standard preferences and the C–CAPM. By contrast $\alpha/\rho = 0$ (*i.e.*, $\gamma = 1$) delivers an intertemporal CAPM, where risk premia are determined by the covariance with the market portfolio. These loci are plotted in Figure 1 (where γ is plotted on the vertical axis against η on the horizontal axis), along with $\eta = 1$ and a fourth locus that we discuss presently.

Recall that the dividend-denominated state–price deflator is given by $\mathcal{G}_d(t) = \mathcal{G}(t)\,c(t)$. In terms of observables (using (4.11)) we have

$$\mathcal{G}_d(t) = e^{-\beta\,(\alpha/\rho)\,t}\,c(t)^{b_2}\,\phi(t)^{b_1},$$

where $b_1 = (\alpha/\rho) - 1$ and $b_2 = 1 - (\alpha/\rho)\,(1-\rho)$. The conditions $b_1 = 0$ and $b_2 = 0$ are equivalent to $\varepsilon = 0$ in Table 1. Here is the connection: When the dividend-denominated state–price deflator depends only on the forcing variable, then so do the dividend-denominated interest rate and price of risk. We consider each case in turn. First, the locus $b_1 = 0$ (*i.e.*, $\alpha/\rho = 1$ or $\gamma\,\eta = 1$) is plotted in Figure 1 as the rectangular hyperbola of standard preferences. In this case, $\varepsilon = 0$ in (4.7) for $y = c$ and $y = m$. For $y = c$, the endowment deflator depends only on consumption: $\mathcal{G}_d(t) = e^{-\beta\,t}\,c(t)^{\rho}$. For $y = m$, since $\mathcal{G} = m$, we can write endowment deflator solely in terms of the state–price deflator: $\mathcal{G}_d(t) = e^{-\beta\,(1-\rho)\,t}\,m(t)^{\rho/(\rho-1)}$. Second, the locus $b_2 = 0$ (*i.e.*, $\gamma + \eta = 2$) is plotted in Figure 1 as the diagonal line. In this case, $\varepsilon = 0$ in (4.7) for $y = \phi$; the endowment deflator depends only on the capital account: $\mathcal{G}_d(t) = \phi(t)^{\rho/(1-\rho)}$. In each case,then, the simplification that $\varepsilon = 0$ achieves can be ascribed to the fact that $\mathcal{G}_d(t)$ can be expressed as a function of the forcing variable.

Solving for ψ when $\rho = 0$. For $y = c$, (2.16) provides a semi-explicit solution to (3.11) when $\rho = 0$. More generally for y,

$$\log(\psi(t)) = \int_{s=t}^{T} \beta\,e^{-\beta\,(s-t)}\,d(t,s)\,ds + e^{-\beta\,(T-t)} d(t,T) + e^{-\beta\,(T-t)}\,\log(\zeta) + \int_{s=t}^{T} e^{-\beta\,(s-t)}\,E_t\left[C(s)\right]ds, \tag{4.13}$$

where

$$d(t,s) = \int_{u=t}^{s} E_t\left[\widetilde{\mu}_y(u)\right]\,du = E_t\left[\log\left(\frac{y(s)}{y(t)}\right)\right]$$

$$C(s) = a_0 + a_2\,\frac{1}{2}\,\|a_1\,\sigma_y(s) + \sigma_\psi(s)\|^2.$$

Applying Itô's lemma to (4.13) produces

$$\widetilde{\mu}_\psi(t) = \beta\,\log(\psi(t)) - a_1\,\widetilde{\mu}_y(t) - C(t), \tag{4.14}$$

thereby confirming that (4.13) is the solution to (3.11), and

$$\sigma_\psi(t) = \int_{s=t}^{T} \beta\,e^{-\beta\,(s-t)}\,\hat{\Sigma}(t,u)\,ds + e^{-\beta\,(T-t)}\,\hat{\Sigma}(t,T) + \int_{s=t}^{T} e^{-\beta\,(s-t)}\,\hat{\sigma}_C(t,s), \tag{4.15}$$

where

$$\hat{\Sigma}(t,s) := \int_{u=t}^{s} \widehat{\sigma}_{\widetilde{\mu}_y}(t,u)\,du$$

and where $\widehat{\sigma}_{\widetilde{\mu}_y}(t,u)$ is the volatility of $E_t[\widetilde{\mu}_y(u)]$ and $\hat{\sigma}_C(t,s)$ is the volatility of $E_t[C(s)]$.[23] If $\widetilde{\mu}_y$ and σ_y are constant, then $\sigma_\psi = 0$ and (4.13) specializes to

$$\log(\psi(t)) = \frac{1}{\beta}\left(1 - e^{-\beta\,(T-t)}\right)\left(a_0 + a_1\,\widetilde{\mu}_y + a_2\,\frac{1}{2}\,\|\sigma_y\|^2\right) + e^{-\beta\,(T-t)}\,\log(\zeta). \tag{4.16}$$

If (4.15) is independent of ψ, then (4.15) and (4.13) provide a fully explicit solution for ψ. This is the case, for example, when $C(t)$ is deterministic, in which case the last term on the right-hand side of (4.15) is zero. Either of the following two conditions ensures that $C(t)$ is deterministic:

Condition 1. $\gamma = 1$.

Condition 2. $\widetilde{\mu}_y(t)$ *is Gaussian and* $\sigma_y(t)$ *is constant.*

Under Condition 1 (which produces log utility), $C(t) = a_0$. Under Condition 2, $\widehat{\sigma}_{\widetilde{\mu}_y}(t,u)$ is a deterministic function of $u - t$, which latter ensures that $\sigma_\psi(t)$ is deterministic for the finite-horizon problem and constant for the infinite-horizon problem.[24]

Now we establish a related result for $\rho \neq 0$ and under Condition 2. We assume $\varepsilon = 0$ so that the solution for $\pi(t)$ is given by (4.2). Given these assumptions, r_d is Gaussian and λ_d is constant, which together imply the weak form of the expectations hypothesis holds for the dividend-denominated term structure. Applying Itô's lemma to π given in (4.2) and using the implication of the weak form given by (B.8), we have

$$\sigma_\psi(t) = a_1\left\{\int_{s=t}^{T} w_d(t,s)\,\hat{\Sigma}(t,u)\,ds + \zeta^\rho\left(\frac{w_d(t,T)}{\beta}\right)\hat{\Sigma}(t,T)\right\}, \tag{4.17}$$

[23] Campbell (1993) derives a similar result when $y = \phi$ for the infinite-horizon problem in a discrete-time version of this model.

[24] Condition 2 is not the most general condition that generates a deterministic $C(t)$.

where $w_d(t,s) := p_d(t,s)/\pi(t)$ and a_1 is given in Table 1, and where we have used the facts that $\hat{\sigma}_{r_d}(t,u) = d_1\, \hat{\sigma}_{\widetilde{\mu}_y}(t,u)$ and $\sigma_\psi = (1/\rho)\, \bar{\sigma}_\pi/\pi$.

Recall that when $y = 1/m$, $\widetilde{\mu}_y(t) = r(t) + \frac{1}{2}\,\|\lambda(t)\|^2$ and $\sigma_y(t) = \lambda(t)$, where r is the real interest rate. Our assumption that σ_y is constant implies that $d\widetilde{\mu}_y(t) = dr(t)$, so that $\widehat{\sigma}_{\widetilde{\mu}_y}(t,u) = \widehat{\sigma}_r(t,u)$. On the other hand, the volatility of a real coupon bond is given by (B.4b) where σ_p is given by (B.8). Assuming an infinite horizon solution exists, we have established $\eta\,\sigma_\psi \approx -\sigma_\varpi$. They differ only by the weights: $w_d(t,s)$ versus $w(t,s) := p(t,s)/\varpi(t)$. However, even when $\eta = 1$, the weights do not converge: When $\eta = 1$, the endowment perpetuity weights are $w_d(t,s) = \beta\, e^{-(s-t)\,\beta}$, while the real perpetuity weights are determined by r and λ and need not bear any particular relation to $w_d(t,s)$. Nevertheless, whenever consumption remains constant we have m_d proportional to m, so that the real and endowment term structures are identical. This situation occurs with $\eta = 0$ and $\gamma = \infty$ and does not require homoskedasticity. (See the discussion on limit cases below.)[25]

The existence of an infinite-horizon solution. The interpretation of π as the value of a fixed income asset provides a framework for analyzing the existence of a utility function in limit as the horizon goes to infinity. Define the dividend-denominated forward rate as $f_d(t,s) := -\partial \log(p_d(t,s))/\partial s$, and define the asymptotic forward rate as $\varphi_d := \lim_{s\to\infty} f_d(t,s)$ when a finite limit exists. Then (i) if $\varphi_d > 0$, an infinite-horizon solution exists; (ii) if $\varphi_d < 0$, an infinite-horizon solution does not exist; (iii) if $\varphi_d = 0$ and $\lim_{s\to\infty} p_d(t,s) \neq 0$, an infinite-horizon solution does not exist; (iv) if $\varphi_d = 0$ and $\lim_{s\to\infty} p_d(t,s) = 0$, more information is needed to determine whether an infinite-horizon solution exists.[26]

When the dividend-denominated interest rate is deterministic, the asymptotic dividend-denominated forward rate is simply $\varphi_d = \lim_{t\to\infty} r_d(t)$, and $\varphi_d = r_d$ for constant r_d. For $\rho = 0$, $r_d = \beta > 0$, and we are guaranteed of having an infinite-horizon solution for all parameter values. For $\rho \neq 0$, let us examine standard preferences at its extremes as an example. Table 2 indicates that (i) $y = c$ is feasible for $(\eta,\gamma) = (\infty, 0)$ and (ii) $y = 1/m$ is feasible for $(\eta,\gamma) = (0,\infty)$. In the first case, $r_d = \beta - \widetilde{\mu}_c - \frac{1}{2}\,\|\sigma_c\|^2$, and we must have $\widetilde{\mu}_c + \frac{1}{2}\|\sigma_c\|^2 < \beta$. In the second case, $r_d = r$, and the real interest rate must be positive.

When the dividend-denominated interest rate is stochastic, as long as $\varepsilon = 0$, we can compute the asymptotic dividend-denominated forward rate from dividend-denominated bond prices given the exogenously specified dividend-denominated interest rate and price of risk. In the more general case when $\varepsilon \neq 0$, neither r_d nor λ_d would be exogenous and we could not directly solve for bond prices. Nevertheless, as shown in Appendix B, if the asymptotic dividend-denominated forward rate were negative, then $\lim_{T\to\infty} \sigma_\pi(t,T) = \lim_{T\to\infty} \sigma_{p_d}(t,T) = \sigma_{p_d}(t,\infty)$ if the limit exists.

[25]Campbell (1993) discusses the relation between the real and endowment perpetuities. We can recover his expression with the change of variables $\omega := \psi^{1/\eta}$, so that $\sigma_\omega \approx -\sigma_\varpi$ instead of $\sigma_\psi \approx -\eta\,\sigma_\varpi$.

[26]See Appendix B.

Therefore, in the region of nonconvergence we can solve the following problem

$$\mu_{p_d}(t,T) = r_d(t) + \lambda_d(t)^\top \sigma_{p_d}(t,T), \tag{4.18}$$

where

$$r_d(t) = d_0 + d_1\,\widetilde{\mu}_y(t) + d_1\,d_2\,\frac{1}{2}\,\|\sigma_y(t)\|^2 - (\varepsilon/d_1)\frac{1}{2}\,\|\sigma_{p_d}(t,\infty)\|^2 \tag{4.19a}$$

$$\lambda_d(t) = -d_2\,\sigma_y(t) + (\varepsilon/d_1)\,\sigma_{p_d}(t,\infty). \tag{4.19b}$$

The correct specification of $\sigma_{p_d}(t,\infty)$ in (4.19) produces an internally consistent solution. We provide an illustration in Section 5.

Limit cases. In this section we examine the model at the limit values of the parameter space for intertemporal substitution and risk aversion. We consider the cases where η and γ are zero or infinity. We can identify certain features of the solution without solving the model entirely. However, care must be taken: For a given limit, there may be no equilibrium for arbitrarily chosen dynamics of a given forcing variable. By examining the limiting values of d_1 and d_2 in Table 2, we can see that certain combinations of forcing processes and limiting parameter values are infeasible (if either d_1 or d_2 goes to infinity). See columns 6–9 of Table 2 for a summary.

Case 1: $\eta = 0$ ($y = \phi$ or $1/m$).

This corresponds to extreme unwillingness to substitute intertemporally. We note that $\lim_{\eta\to 0}\ u(x) = 0$ and $\lim_{\eta\to 0}\ \psi = \lim_{\eta\to 0}\ (\beta\,\pi)^{\eta/(\eta-1)} = 1$. In (2.8), these restrictions impose a restriction on the consumption process: $\widetilde{\mu}_c = (\gamma-1)\,\frac{1}{2}\,\|\sigma_c\|^2$. If $\gamma = 1$, the CAPM case, then log consumption is a martingale.

Case 2: $\eta = \infty$ ($y = c$).

This corresponds to a perfect willingness to substitute intertemporally. This value of η forces a restriction on the process for the state–price deflator that can be expressed as $r + \frac{1}{2}\,\|\lambda\|^2 = \beta$ and $\lambda = \gamma\,\sigma_\phi$. (Solve (2.13) and (3.9).)

Case 3: $\gamma = 0$ ($y = c$ or ϕ).

This is the case of risk neutrality with respect to static wealth gambles. However, as we have shown above, agents are not risk neutral with respect to dynamic consumption processes. Although we have ruled out $y = 1/m$ in this case, there is in fact a solution; but in general it involves setting $c(t) = 0$ often, a corner solution that does not satisfy our system of equations (since we have assumed the strict positivity of consumption).

Case 4: $\gamma = \infty$ ($y = 1/m$).

In this case the agent is extremely risk-averse with respect to static wealth gambles. See Case 6.

Case 5: $\eta = 0$ and $\gamma = 0$ ($y = \phi$).

In this case the agent is unwilling to substitute consumption across periods but perfectly willing to substitute across states of nature. The state-price deflator is $m(t) = c(t)/\phi(t)$, and from (2.8), we infer that $c(t)$ is a martingale, a fact that has far-reaching implications in the infinite-horizon case. Because $c(t)$ is a positive martingale, it converges. If technology is such that a constant and strictly positive asymptotic consumption flow is feasible, then consumption might converge to such a value. Otherwise, the consumer asymptotically exhausts his wealth and consumption converges to zero, for any value of the rate of time preference β. For example, suppose that μ_ϕ is constant (even if $\widetilde{\mu}_\phi$ and σ_ϕ are not). Then the solution is $1/\pi(t) = \psi(t) = r(t) = \mu_\phi$, $\lambda(t) = 0$, and $\sigma_c(t) = \sigma_k(t) = \sigma_\phi(t)$. If $\sigma_\phi(t)$ does not go to zero, then $c(t)$ and $k(t)$ both do almost surely (though, of course, not in the L_1 norm).

Case 6: $\eta = 0$ and $\gamma = \infty$ ($y = 1/m$).

Given the results in Case 1, (2.8) can only be satisfied if $\widetilde{\mu}_c = \widetilde{\sigma}_c = 0$, which means that $c(t)$ is a constant determined by the initial wealth. Not surprisingly (3.10) shows that the optimal portfolio is determined entirely by the hedging component: $\sigma_\phi(t) = -\sigma_\psi(t)$. Since consumption is constant, $r_d = r$ and $\lambda_d = \lambda$, so $\pi(t)$ is the value of a real coupon bond as well as that of a dividend-denominated coupon bond. Finally, given $\pi(t) = 1/(\beta\,\psi(t))$ when $\eta = 0$, we have $-\sigma_\psi(t) = \sigma_\pi(t)$. Therefore the optimal portfolio is a real annuity in this case (a real perpetuity in the infinite-horizon case).[27]

Case 7: $\eta = \infty$ and $\gamma = 0$ ($y = c$).

This is the case of risk-neutrality. The price of risk vanishes and, from (4.12a), the interest rate is determined by the rate of time preference, $r(t) = \beta$. As a result, the expected rate of return on any asset is equal to β, and the yield curve is flat at that level.

Case 8: $\eta = \infty$ and $\gamma = \infty$.

This combination is not feasible with any of the three forcing processes.

5. Illustrations and Markovian structure

In this section we provide some illustrations with closed-form solutions. First, we introduce a Markovian structure that we will use for the examples and for the numerical solutions in Section 6.

Markovian structure. We suppose there are d Markovian state variables X driving the forcing process y, where $y = c$, or $1/m$, or ϕ. The joint dynamics of X and y are given by

$$\begin{pmatrix} dX(t) \\ d\log(y(t)) \end{pmatrix} = \begin{pmatrix} \mu_X(X(t)) \\ \widetilde{\mu}_Y(X(t)) \end{pmatrix} dt + \begin{pmatrix} \sigma_X(X(t))^\top \\ \sigma_Y(X(t))^\top \end{pmatrix} dW(t),$$

where $W^\top = \left(W_x^\top, W_y\right)$ is a l-dimensional vector of orthonormal Brownian motions, with W_x $(l-1)$-dimensional and W_y scalar. The dimensions of $\sigma_X(x)$ and $\sigma_Y(x)$ are respectively $l \times d$ and $l \times 1$. We assume that the last column of $\sigma_X(x)^\top$ is a vector of zeros, so that the state variables are not affected by W_y: $\sigma_X(x)^\top = \left(\Sigma_X(x)^\top\ 0\right)$, where $\Sigma_X(x)$ is $(l-1) \times d$.[28] Note that even if the state variables are deterministic ($\sigma_X(x) \equiv 0$), y can be stochastic; a completely deterministic economy

[27] This point is made by Campbell and Viceira (1996).

[28] This assumption is without loss of generality. It simply allows for the possibility that there exists a shock, W_y, that affects y but not X.

would require $\sigma_Y(x) \equiv 0$ as well. The following functions of the state variables are called collectively *the data*:

$$\mu_X(x), \quad \widetilde{\mu}_Y(x), \quad \sigma_X(x)^\top \sigma_X(x), \quad \sigma_X(x)\,\sigma_Y(x), \quad \text{and} \quad \sigma_Y(x)^\top \sigma_Y(x). \tag{5.1}$$

For the most part, we will take $\zeta > 0$ so that the information variable is strictly positive and finite. In addition, since we have assumed the data do not depend on time, the solution will be time-homogeneous. As a consequence, it is convenient to model $\log(\psi(t)) = \Omega(X(t), T-t)$. Given the function $\Omega(x,\tau)$, we can compute the functions for the drift and diffusion $\widetilde{\mu}_\psi(t) = \mu_\Omega(X(t), t)$ and $\sigma_\psi(t) = \sigma_\Omega(X(t), t)$, in terms of the partial derivatives of Ω:

$$\mu_\Omega(x,\tau) = \Omega_x(x,t)^\top \mu_X(x) + \frac{1}{2}\,\mathrm{tr}\left[\sigma_X(x)^\top \sigma_X(x)\,\Omega_{xx}(x,\tau)\right] - \Omega_\tau(x,\tau) \tag{5.2a}$$

$$\sigma_\Omega(x,\tau) = \Omega_x(x,\tau)\,\sigma_X(x). \tag{5.2b}$$

The data turn (3.11) into a quasi-linear partial differential equation (PDE) in terms of the unknown function Ω,

$$a_0 + a_1\,\widetilde{\mu}_Y(x) + \mu_\Omega(x,\tau) + a_2\,\frac{1}{2}\,\|a_1\,\sigma_Y(x) + \sigma_\Omega(x,\tau)\|^2 + \beta\,u\left(e^{-\Omega(x,\tau)}\right) = 0, \tag{5.3}$$

subject to the boundary condition $\Omega(x,0) = \log(\zeta)$.[29] If the data are real analytic, the Cauchy–Kowaleskaya theorem guarantees a unique real analytic function $\Omega(x,\tau)$ exists in the neighborhood of $\tau = 0$.[30] The theorem does not guarantee the existence of a solution for an arbitrary finite horizon. Below, we provide an example that fails to have such a solution. However, if solutions exist for all finite horizons, then they converge to an infinite-horizon solution if and only if $\lim_{\tau\to\infty}\ \Omega_\tau(x,\tau) = 0$. In Section 4 we examined the conditions for an infinite-horizon solution to exist, and we provide illustrations in this section.

A single state variable. For the illustrations, we entertain three models, each with a single affine state variable.

Model 1 (this model is Gaussian-affine):

$$dx = \kappa\,(\bar{x} - x)\,dt + s_X\,dW_1 \tag{5.4a}$$

$$d\log(y) = (a_y + b_y\,x)\,dt + s_1\,dW_1 + s_2\,dW_2. \tag{5.4b}$$

Model 2 (this model is "square-root"-affine):

$$dx = \kappa\,(\bar{x} - x)\,dt + s_X\,\sqrt{x}\,dW_1 \tag{5.5a}$$

$$d\log(y) = (a_y + b_y\,x)\,dt + s_1\,\sqrt{x}\,dW_1 + s_2\,\sqrt{x}\,dW_2. \tag{5.5b}$$

Model 3 (this model is Gaussian-quadratic):

$$dx = \kappa\,(\bar{x} - x)\,dt + s_X\,dW_1 \tag{5.6a}$$

$$d\log(y) = \left(a_y + b_y\,x + c_y\,x^2\right)dt + (s_1 + s_{1X}\,x)\,dW_1 + (s_2 + s_{2X}\,x)\,dW_2, \tag{5.6b}$$

[29] Duffie and Lions (1992) address the existence and uniqueness of PDE solutions in a related setting.

[30] See Rauch (1991, Chapter 1), for example.

The parameters κ, $\bar{x}$, s_X, a_y, b_y, c_y, s_1, s_{1X}, s_2, and s_{2X} are scalar constants.

Example 1: Constant dividend-denominated interest rate. Here we adopt (5.4). If $b_y = 0$, (4.7a) gives $r_d = d_0 + d_1 \left(a_y + d_2 \frac{1}{2} (s_1^2 + s_2^2)\, \bar{x}\right)$. For example, if $y = 1/m$, then

$$r_d = \eta\, \beta + (1 - \eta) \left(r + \frac{\|\lambda\|^2}{2\, \gamma} \right), \tag{5.7}$$

where $\|\lambda\|^2 = (s_1^2 + s_2^2)\, \bar{x}$. In addition, for $\rho = 0$ (4.16) gives

$$\log(\psi(t)) = \frac{1}{\beta} \left(1 - e^{-(T-t)\, \beta}\right) \left(r - \beta + \frac{\|\lambda\|^2}{2\, \gamma} \right) + e^{-\beta\, (T-t)} \log(\zeta). \tag{5.8}$$

For all parameter values we have $\sigma_\pi = \sigma_\psi = 0$. As a consequence, the optimal portfolio is mean–variance efficient and the dynamics of optimal consumption and the optimal portfolio weights are independent of the horizon. Optimal consumption dynamics (which are given by the solution to (2.13)) are

$$\widetilde{\mu}_c = \eta\, (r - \beta) + \left(\frac{1}{\gamma} - \frac{1 - \gamma\, \eta}{\gamma^2} \right) \frac{1}{2}\, \|\lambda\|^2 \quad \text{and} \quad \sigma_c = \frac{\lambda}{\gamma}. \tag{5.9}$$

Another consequence is $\|\sigma_c\|^2 = \|\sigma_\phi\|^2$ (see (3.9b)), which is the main difficulty this model has with the equity premium puzzle data. In order to get $\|\sigma_c\|^2 \ll \|\sigma_\phi\|^2$, sufficient state variation must be introduced.

Example 2: Solving for ψ given $\rho = 0$, part 1. We continue to adopt (5.4), but we allow $b_y \neq 0$ and treat the infinite-horizon case. In this example, we use (4.13) and (4.15) to solve for $\log(\psi)$ given $\rho = 0$. The conditional expectation of x and its volatility are given by

$$\begin{aligned} E_t[x(u)] &= \bar{x} + e^{-\kappa\, (u-t)}\, (x(t) - \bar{x}) \\ \widehat{\sigma}_x(t, u) &= s_X\, e^{-\kappa\, (u-t)}, \end{aligned}$$

(we consistently drop the second component of the volatilities, because they all vanish). The conditional expectation of $\widetilde{\mu}_y$ and its volatility are given by $E_t[\widetilde{\mu}_y(u)] = a_y + b_y\, E_t[x(u)]$ and $\widehat{\sigma}_{\widetilde{\mu}_y}(t, u) = b_y\, \widehat{\sigma}_x(t, u)$. Thus

$$d(t, s) = (s - t)\, (a_y + b_y\, \bar{x}) + \left(\frac{1 - e^{-\kappa\, (s-t)}}{\kappa} \right) b_y\, (x(t) - \bar{x}) \tag{5.10a}$$

$$\hat{\Sigma}(t, s) = b_y\, s_X \left(\frac{1 - e^{-\kappa\, (s-t)}}{\kappa} \right). \tag{5.10b}$$

Using (5.10a),

$$\int_{s=t}^{\infty} \beta\, e^{-\beta\, (s-t)}\, d(t, s)\, ds = \frac{a_y + b_y\, \bar{x}}{\beta} + \frac{b_y\, (x(t) - \bar{x})}{\beta + \kappa},$$

which can be interpreted as the sum of permanent and transitory components. Using (4.15) and (5.10b),

$$\sigma_\psi = \int_{s=t}^{\infty} \beta\, e^{-\beta\,(s-t)}\, \hat{\Sigma}(t,s)\, ds = \frac{b_y\, s_X}{\beta+\kappa}. \tag{5.11}$$

Given (5.11) we can write

$$C = a_0 + a_2\, \frac{1}{2} \left\{ \left(s_1 + \frac{b_y\, s_X}{\beta+\kappa} \right)^2 + s_2^2 \right\}.$$

Note that C is state-independent, so that the calculation of σ_ψ in (5.11) is correctly based on (4.15), which therefore has the solution:

$$\log(\psi) = \frac{b_y\,(x-\bar{x})}{\beta+\kappa} + \frac{a_y + b_y\,\bar{x} + C}{\beta}. \tag{5.12}$$

The first-order approximation for the wealth–consumption ratio around $\rho = 0$ is $\pi = 1/\beta + (\rho/\beta)\, \log(\psi) + \mathcal{O}(\rho^2)$. If we take $y = \phi$ and set $a_y = 0$ and $b_y = 1$, we have the setting adopted by Campbell (1993) and Campbell and Koo (1997). In Section 6 we apply our numerical solution method to solve this model over the entire parameter space.

Example 3: Solving for ψ given $\rho = 0$, part 2. In the previous example, the state-independence of σ_ψ allowed us to compute it directly from (4.15). In this example, we take the PDE approach to obtain the same solution. Then we apply the PDE approach to models (5.4) and (5.5), where the state-dependence makes the direct approach more difficult.

First we attack (5.4). In this case, the solution can be written as $\Omega(x,\tau) = \delta_0(\tau) + \delta_1(\tau)\,(x-\bar{x})$. PDE (5.3) decomposes into a pair of ODEs:

$$\begin{aligned} \delta_0'(\tau) &= \chi_0 + \chi_1\, \delta_1(\tau) + \chi_2\, \delta_1(\tau)^2 - \beta\, \delta_0(\tau) \\ \delta_1'(\tau) &= \theta_0 + \theta_1\, \delta_1(\tau), \end{aligned}$$

subject to $\delta_0(0) = \log(\zeta)$ and $\delta_1(0) = 0$, where

$$\begin{aligned} \chi_0 &= a_2\, \frac{1}{2}\, (s_1^2 + s_2^2) + a_0 + a_y + b_y\, \bar{x} \\ \chi_1 &= a_2\, s_1\, s_X \\ \chi_2 &= a_2\, \frac{1}{2}\, s_X^2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \theta_0 &= b_y \\ \theta_1 &= -(\beta+\kappa). \end{aligned}$$

The solutions are

$$\delta_0(\tau) = e^{-\beta\,\tau}\, \log(\zeta) + \chi_0 \left(\frac{1-e^{-\beta\,\tau}}{\beta} \right) + \int_{s=0}^{\tau} e^{-\beta\,(\tau-s)} \left(\chi_1\, \delta_1(s) + \chi_2\, \delta_1(s)^2 \right) ds \tag{5.13a}$$

$$\delta_1(\tau) = \frac{b_y \left(1 - e^{-(\beta+\kappa)\,\tau}\right)}{\beta+\kappa}. \tag{5.13b}$$

For the infinite-horizon, (5.13) specializes to (5.12).

Now we attack (5.5). Again, $\Omega(x,\tau) = \delta_0(\tau) + \delta_1(\tau)\,(x-\bar{x})$ and PDE (5.3) decomposes into

$$\begin{aligned}\delta_0'(\tau) &= \chi_0 + \chi_1\,\delta_1(\tau) + \chi_2\,\delta_1(\tau)^2 - \beta\,\delta_0(\tau)\\ \delta_1'(\tau) &= \theta_0 + \theta_1\,\delta_1(\tau) + \theta_2\,\delta_1(\tau)^2,\end{aligned}$$

subject to $\delta_0(0) = \log(\zeta)$ and $\delta_1(0) = 0$, where

$$\begin{aligned}\chi_0 &= a_2\,\frac{1}{2}\,(s_1^2 + s_2^2) + a_0 + a_y + b_y\,\bar{x}\\ \chi_1 &= a_2\,s_1\,s_X\\ \chi_2 &= a_2\,\frac{1}{2}\,s_X^2\end{aligned} \qquad \text{and} \qquad \begin{aligned}\theta_0 &= a_2\,\frac{1}{2}\,(s_1^2 + s_2^2) + b_y\\ \theta_1 &= a_2\,s_1\,s_X - (\beta+\kappa)\\ \theta_2 &= a_2\,\frac{1}{2}\,s_X^2.\end{aligned}$$

The solutions are

$$\delta_0(\tau) = e^{-\beta\,\tau}\,\log(\zeta) + \chi_0\left(\frac{1-e^{-\beta\,\tau}}{\beta}\right) + \int_{s=0}^{\tau} e^{-\beta\,(\tau-s)}\left(\chi_1\,\delta_1(s) + \chi_2\,\delta_1(s)^2\right)ds \tag{5.14a}$$

$$\delta_1(\tau) = \frac{2\,\theta_0\,\theta_1\left(e^{\Delta\,\tau}-1\right)}{\theta_2\left(\theta_1\,(1-e^{\Delta\,\tau}) + \Delta\,(1+e^{\Delta\,\tau})\right)}, \quad \text{where } \Delta = \sqrt{{\theta_1}^2 - 4\,\theta_0\,\theta_2}. \tag{5.14b}$$

Now we adopt (5.6). In this case $\Omega(x,\tau) = \delta_0(\tau) + \delta_1(\tau)\,(x-\bar{x}) + \delta_2(\tau)\,(x-\bar{x})^2$ and PDE (5.3) now decomposes into three ODEs:

$$\begin{aligned}\delta_0'(\tau) &= \chi_0 + \chi_1\,\delta_1(\tau) + \chi_2\,\delta_1(\tau)^2 + \chi_3\,\delta_2(\tau) - \beta\,\delta_0(\tau)\\ \delta_1'(\tau) &= \theta_0 + \theta_1\,\delta_1(\tau) + \theta_2\,\delta_2(\tau) + \theta_3\,\delta_1(\tau)\;\delta_2(\tau)\\ \delta_2'(\tau) &= \epsilon_0 + \epsilon_1\,\delta_2(\tau) + \epsilon_2\,\delta_2(\tau)^2,\end{aligned}$$

subject to $\delta_0(0) = \log(\zeta)$ and $\delta_1(0) = \delta_2(0) = 0$, where χ_0, χ_1, and χ_2 are as given above, $\chi_3 = 2\,s_X^2\,\bar{x}$, and

$$\begin{aligned}\theta_0 &= b_y + 2\,c_y\,\bar{x}\\ \theta_1 &= -(\beta+\kappa)\\ \theta_2 &= 2\,a_2\,s_1\,s_X\,\bar{x}\\ \theta_3 &= 2\,a_2\,s_X^2\,\bar{x}\end{aligned} \qquad \text{and} \qquad \begin{aligned}\epsilon_0 &= c_y + a_2\,\frac{1}{2}\left(s_1^2 + s_2^2\right)\\ \epsilon_1 &= -(\beta + 2\,\kappa) + 2\,a_2\,s_1\,s_X\\ \epsilon_2 &= 2\,a_2\,s_X^2.\end{aligned}$$

The solutions are

$$\begin{aligned}\delta_0(\tau) = e^{-\beta\,\tau}\,\log(\zeta) + \chi_0\left(\frac{1-e^{-\beta\,\tau}}{\beta}\right) &+\\ \int_{s=0}^{\tau} e^{-\beta\,(\tau-s)}&\left(\chi_1\,\delta_1(s) + \chi_2\,\delta_1(s)^2 + \chi_3\,\delta_2(s)\right)ds\end{aligned} \tag{5.15a}$$

$$\delta_1(\tau) = \int_{s=0}^{\tau} \exp\left\{\theta_1\,(\tau-s) + \theta_3\int_{u=s}^{\tau}\delta_2(u)\,du\right\}(\theta_0 + \theta_2\,\delta_2(s))\,ds \tag{5.15b}$$

$$\delta_2(\tau) = \frac{2\,\epsilon_0\,\epsilon_1\left(e^{\Delta\,\tau}-1\right)}{\epsilon_2\left(\epsilon_1\,(1-e^{\Delta\,\tau}) + \Delta\,(1+e^{\Delta\,\tau})\right)}, \quad \text{where } \Delta = \sqrt{{\epsilon_1}^2 - 4\,\epsilon_0\,\epsilon_2}. \tag{5.15c}$$

Term structure example. As a specific example, we take the consumption process as given and we solve for the term structure of interest rates, paralleling Duffie and Epstein (1992a) and Duffie, Schroder, and Skiadas (1997). Those papers treat the inhomogeneous case, while we treat the homogeneous case.[31]

Using (2.13), the interest rate and price of risk are given by

$$r = \beta + \widetilde{\mu}_c + \left(\alpha - \frac{1}{2}\right) \|\sigma_c\|^2 + \alpha\, \sigma_c^\top \sigma_\psi \tag{5.16a}$$

$$\lambda = (1-\alpha)\, \sigma_c - \alpha\, \sigma_\psi \tag{5.16b}$$

for $\rho = 0$. In order to compute the term structure, we need

$$\sigma_\psi = \begin{pmatrix} \sigma_X\, \Omega_x(x,\tau) \\ 0 \end{pmatrix},$$

where

$$\sigma_X\, \Omega_x(x,\tau) = \begin{cases} s_X\, \delta_1(\tau) & \text{for Model 1} \\ s_X\, \delta_1(\tau)\, x & \text{for Model 2} \\ s_X \left(\delta_1(\tau) + \delta_2(\tau)\, x\right) & \text{for Model 3,} \end{cases}$$

where the factor loadings are specific to each model. We see that r and λ depend on the horizon τ in the finite-horizon setting unless $\alpha = 0$ (log utility).

Duffie and Epstein (1992a) adopt Model 2 and set $a_y = -\beta$. Although they use a inhomogeneous terminal reward, (5.16) agrees with their term structure in the time-independent case (either $T = \infty$ or $\alpha = 0$): the term structure of Cox, Ingersoll, Jr., and Ross (1985b). Duffie, Schroder, and Skiadas (1997) solve for a Gaussian term structure (with inhomogeneous utility) in a setting where they can vary the way in which uncertainty is resolved. Model 1 is in the spirit of their paper. Let us examine the infinite-horizon problem with that setup: $r = r_0 + r_1\, x$ and $\lambda = (\lambda_1, \lambda_2)^\top$, where

$$\begin{aligned} r_0 &= \beta + a_y + \left(\alpha - \frac{1}{2}\right)\left(s_1^2 + s_2^2\right) + \frac{\alpha\; s_1\, s_X\, b_y}{\beta + \kappa} \\ r_1 &= b_y \\ \lambda_1 &= (1-\alpha)\, s_2 - \frac{\alpha\, s_X\, b_y}{\beta + \kappa} \\ \lambda_2 &= (1-\alpha)\, s_2. \end{aligned}$$

Real bond prices are of the form $P(x,\tau) = \exp(A(\tau) + B(\tau)\, x)$, where

$$B(\tau) = \frac{b_y\, (e^{-\kappa\, \tau} - 1)}{\kappa}.$$

Let note that $\lim_{\tau\to\infty}\, B(\tau) = b_y/\kappa$. The asymptotic forward rate is

$$\varphi = r_0 + r_1\, \bar{x} - \left(\frac{r_1\, s_X\, \lambda_1}{\kappa} + \frac{1}{2}\left(\frac{r_1\, s_X}{\kappa}\right)^2\right).$$

[31] We discuss the relation between the two cases in Appendix C.

Therefore,

$$\frac{\partial \varphi}{\partial \alpha} = s_1^2 + s_2^2 + \frac{(b_y\, s_X)^2}{\kappa\,(\beta+\kappa)} + b_y\, s_X \left(\frac{s_2}{\beta+\kappa} + \frac{s_2}{\kappa} \right),$$

the sign of which depends on the covariance in the last term on the right-hand side. A similar computation can easily be made for Model 3.

Example 4: Regions of convergence. Here we compute the boundary for the regions of convergence. Consider model (5.4). If dividend-denominated bond prices are exponential-affine, where $P_d(x,\tau) = \exp(A(\tau) + B(\tau)\, x)$, then $\sigma_{p_d}(t,\infty) = (s_X\, B(\infty), 0)$, and (4.7) implies $r_d = r_0 + r_1\, x$ and $\lambda_d = (\lambda_1, \lambda_2)^\top$ where

$$\begin{aligned} r_0 &= d_0 + d_1\, a_y + \frac{1}{2}\left(d_1\, d_2\,(s_1^2+s_2^2) - (\varepsilon/d_1)\, s_X^2\, B(\infty)^2\right) \\ r_1 &= d_1\, b_y \\ \lambda_1 &= -d_2\, s_1 + (\varepsilon/d_1)\, s_X\, B(\infty) \quad \text{and} \quad \lambda_2 = -d_2\, s_2. \end{aligned}$$

Given the risk-return condition (4.18), the factor loadings comprise a system of ODEs:

$$\begin{aligned} A'(\tau) &= -r_0 + (\kappa\,\bar{x} - \lambda_1\, s_X)\, B(\tau) + \frac{1}{2}\, s_X^2\, B(\tau)^2 \\ B'(\tau) &= -r_1 - \kappa\, B(\tau). \end{aligned}$$

If an asymptotic forward rate exists, the ODEs imply an algebraic system:

$$\begin{aligned} -\varphi_d &= -r_0 + (\kappa\,\bar{x} - \lambda_1\, s_X)\, B(\infty) + \frac{1}{2}\, s_X^2\, B(\infty)^2 \\ 0 &= -r_1 - \kappa\, B(\infty), \end{aligned}$$

the solution to which is $B(\infty) = r_1/\kappa$ and

$$\varphi_d = d_0 + d_1\{a_y + b_y\,\bar{x}\} + d_1\, d_2\, \frac{1}{2}\left\{ \frac{b_y^2\, s_X^2 + 2\,\kappa\, b_y\, s_X\, s_1 + \kappa^2\,(s_1^2+s_2^2)}{\kappa^2} \right\}. \tag{5.17}$$

Recall that nonconvergence requires $\varphi_d \le 0$. Therefore the boundary between regions of convergence and nonconvergence is given by $\varphi_d = 0$. Notwithstanding the ill behavior of the scale of the economy in terms of the wealth–consumption ratio, the dynamics of consumption growth and of the state–price deflator (*i.e.*, the interest rate and the price of risk) are well-behaved as the horizon goes to infinity in the nonconvergent case: They depend only on σ_ψ which does converge.

For models (5.5) and (5.6), we can follow the same steps, although the algebraic system will be quadratic for these models. The correct solution is the limit of the solution to the system of ODEs.

Existence: An example. Although the Cauchy–Kowaleskaya theorem guarantees a unique solution exists in the neighborhood of $\tau = 0$, it does not guarantee the existence of a solution for all finite τ. The absence of finite-horizon solutions will be indicated by the absence of an asymptotic dividend-denominated forward rate. Here we provide an example to illustrate this issue.

Let the forcing variable be $1/m$ and consider Model 2. Let

$$r = x \qquad \text{and} \qquad \lambda = \sqrt{x}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix},$$

so that $\widetilde{\mu}_y = x + \frac{1}{2}\,x\,(s_1^2 + s_2^2)$. Solving for the asymptotic dividend-denominated forward rate produces two solutions, both of which involve $\sqrt{K}$, where

$$K = \gamma^2\,\kappa^2 - (1-\gamma)\;s_X\;\left(2\,\gamma\,\kappa\,s_1 + \left(\gamma\;\left(2 + {s_1}^2\right) + (1-\gamma)\;{s_2}^2\right)\;s_X\right).$$

If $K < 0$, then no asymptotic (dividend-Denominated) forward rate exists. For $\gamma = 1$, $K = \kappa^2$, while for $\gamma = 0$, $K = -s_1^2\,s_X^2$.

Example 5: Optimal portfolio. In this section, we provide two related illustrations. For the first illustration, we take $y = 1/m$ and adopt Model 3, so that

$$r = r_0 + r_1\,x + r_2\,x^2 \quad \text{and} \quad \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} s_1 + s_{1X}\,x \\ s_2 + s_{2X}\,x \end{pmatrix},$$

where

$$r_0 = a_y - \frac{s_1^2 + s_2^2}{2}, \quad r_1 = b_y - (s_1\,s_{1X} + s_2\,s_{2X})\,, \quad \text{and} \quad r_2 = c_y - \frac{s_{1X}^2 + s_{2X}^2}{2}.$$

This setup produces an exponential-quadratic term structure (assuming $r_2 \neq 0$) similar to Constantinides (1992):

$$p(t, t+\tau) = P(x, \tau) = \exp\left\{B_0(\tau) + B_1(\tau)\,x + B_2(\tau)\,x^2\right\},$$

where τ is the maturity of the bond and $B_i(\tau)$ are factor loadings that can be computed in closed form. Using (2.13b), the volatility of optimal consumption is given by $\sigma_c = \gamma^{-1}\,\lambda + \left(1 - (\eta\,\gamma)^{-1}\right)\sigma_\psi$, where $\sigma_\psi = s_X\,\Omega_x(x, \bar{\tau})$ and $\bar{\tau} := T - t$ is the planning horizon. (In this partial equilibrium problem, T is a variable that is private to the consumer; in the economy the price system can price cash flows beyond T.)

Turning to the optimal portfolio, suppose that the consumer must invest his wealth in the money market account, a zero-coupon bond of fixed maturity s (choosing $s > T$ obviates the need to change investment vehicle) and a non-dividend paying stock S. Arbitrage free asset returns are given by

$$\begin{aligned}
\frac{db}{b} &= r\,dt \\
\frac{dS}{S} &= \mu_S\,dt + \sigma_{S_1}\,dW_1 + \sigma_{S_2}\,dW_2 \\
\frac{dp(t,s)}{p(t,s)} &= \mu_p(\tau)\,dt + \sigma_p(\tau)\,dW_1,
\end{aligned}$$

where σ_{S_i} are parameters, $\sigma_p(\tau) = s_X\,(B_1(\tau) + 2\,B_2(\tau)\,x)$, $\tau := s - t$, and

$$\begin{aligned}
\mu_S &= r + (s_1 + s_{1X}\,x)\,\sigma_{S_1} + (s_2 + s_{2X}\,x)\,\sigma_{S_2} \\
\mu_p(\tau) &= r + (s_1 + s_{1X}\,x)\,\sigma_p(\tau)
\end{aligned}$$

The building blocks for the portfolio weights are

$$\Sigma_\phi = \begin{pmatrix} \sigma_{S_1} & \sigma_p(\tau) \\ \sigma_{S_2} & 0 \end{pmatrix} \quad \text{and} \quad \sigma_\psi = \begin{pmatrix} s_X\, \Omega(x, \bar\tau) \\ 0 \end{pmatrix}.$$

As long as the matrix σ_ϕ has full rank, markets are complete. The portfolio weights themselves are

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{1}{\gamma} \begin{pmatrix} \dfrac{s_2 + s_{2X}\, x}{\sigma_{S_2}} \\ \dfrac{s_1 + s_{1X}\, x}{\sigma_p(\tau)} - \dfrac{(s_2 + s_{2X}\, x)\, \sigma_{S_1}}{\sigma_p(\tau)\, \sigma_{S_2}} \end{pmatrix} + \frac{1-\gamma}{\eta\,\gamma} \begin{pmatrix} 0 \\ \dfrac{\Omega_x(x, \bar\tau)}{B_1(\tau) + B_2(\tau)\, x} \end{pmatrix}.$$

In this example, all of the hedging is done with the bond. Moreover, if $\lambda_2 = 0$, then the stock is not held at all (and optimal consumption does not depend on W_2). If the price of risk is zero and preferences are additive, then only the bond is held to hedge the opportunity set and finance optimal consumption which in this case is deterministic.

For the second illustration, we treat the incomplete-market case studied by Campbell and Viceira (1996). The interest rate be constant, $r = r_0$. In addition to the money market account, there is a single risky stock, the dynamics of which are, as before, $dS/S = \mu_S\, dt + \sigma_{S_1}\, dW_1 + \sigma_{S_2}\, dW_2$. In this case $\mu_S = r_0 + l_0 + l_1\, x$ is given, and the dynamics of x are Gaussian as in (5.6a). Referring to (3.17), the price of risk and the portfolio weights are given by

$$\lambda = \frac{l_0 + l_1\, x}{\sigma_{S_1}^2 + \sigma_{S_2}^2} \begin{pmatrix} \sigma_{S_1} \\ \sigma_{S_2} \end{pmatrix} + \left(\frac{1-\gamma}{\eta}\right) \frac{\sigma_{S_2}\, s_X\, \Omega_x(x, \bar\tau)}{\sigma_{S_1}^2 + \sigma_{S_2}^2} \begin{pmatrix} -\sigma_{S_2} \\ \sigma_{S_1} \end{pmatrix} \tag{5.18}$$

and[32]

$$w_1 = \left(\frac{1}{\gamma}\right) \frac{l_0 + l_1\, x}{\sigma_{S_1}^2 + \sigma_{S_2}^2} + \left(\frac{1-\gamma}{\eta\,\gamma}\right) \frac{s_X\, \Omega_x(x, \bar\tau)\, \sigma_{S_1}}{\sigma_{S_1}^2 + \sigma_{S_2}^2}.$$

Inserting $\widetilde\mu_Y = r_0 + \frac{1}{2}\,\|\lambda\|^2$ and $\sigma_Y = \lambda$ (where λ is given in (5.18)) into (5.3) provides the equation to solve. The price system is endogenous unless $\gamma = 1$. For log utility ($\eta = \gamma = 1$), there is a closed-form solution.

6. Numerical solution method

In this section, we present a method for numerically solving (5.3). Our numerical solution technique can be thought of as the method of undetermined coefficient functions. It is based on the exact solutions described in the previous section for $\rho = 0$. To illustrate our method and to reduce the notational burden, we suppose there is a single state variable. Given real analytic data, $\Omega(x, \tau)$ has the power-series representation expanding around $x = x_0$ and treating τ as a parameter:

[32] An equivalent expression appears in Campbell and Viceira (1996). Their expression can be obtained by (i) setting $l_0 = \frac{1}{2}\left(\sigma_{S_1}^2 + \sigma_{S_2}^2\right)$ and $l_1 = 1$ so that $\widetilde\mu_S = r_0 + x$ and (ii) replacing $\sigma_\psi = s_X\, \Omega(x, \bar\tau)$ with σ_π/ρ.

$\Omega(x,\tau) = \sum_{n=0}^{\infty} \delta_n(\tau)\,(x-x_0)^n$. The condition for convergence to an infinite-horizon solution is $\Omega_\tau(x,\tau)=0$. The partial derivatives are given by

$$\Omega_\tau(x,\tau) = \sum_{n=0}^{\infty} \delta_n'(\tau)\,(x-x_0)^n$$

$$\Omega_x(x,\tau) = \sum_{n=1}^{\infty} n\,\delta_n(\tau)\,(x-x_0)^{n-1}$$

$$\Omega_{xx}(x,\tau) = \sum_{n=2}^{\infty} n\,(n-1)\,\delta_n(\tau)\,(x-x_0)^{n-2}.$$

The boundary condition, $\Omega(x,0)=\log(\zeta)$, implies $\delta_0(0)=\log(\zeta)$ and $\delta_i(0)=0$ for $i\geq 1$. The solution method becomes operational by approximating $\Omega(x,\tau)$ as

$$\Omega^N(x,\tau) := \sum_{n=0}^{N} \delta_n^N(\tau)\,(x-x_0)^n,$$

which is inserted into (5.2) and the result is inserted into (5.3), upon which the N-th order Taylor approximation is computed. The result can be separated into a system of nonlinear ordinary differential equations. In the previous section with $\rho=0$, we saw three examples where this representation provided exact solutions with finite N.

For comparison with Campbell (1993) and Campbell and Koo (1997), we treat the case where the forcing variable is the return on optimally invested wealth. We adopt the dynamics given in (5.4). In order to directly compare with their results, we adopt a change of variables. Define $\omega := \psi^{1/\eta}$. We can write (3.11) in terms of ω for $y=\phi$:

$$\beta + \eta\left(\widetilde{\mu}_\phi + \widetilde{\mu}_\omega + (1-\gamma)\,\frac{1}{2}\,\|\sigma_\omega + \sigma_\phi\|^2\right) + \beta\,u(1/\omega^\eta) = 0, \tag{6.1}$$

subject to $\omega(T)=\zeta^{1/\eta}$. Now we let $\log(\omega(t)) = \Omega(X(t),T-t)$. Given (5.4) we have

$$\mu_\Omega(x) = \kappa\,(\bar{x}-x)\,\Omega_x(x,\tau) + s_X^2\,\frac{1}{2}\,\Omega_{xx}(x,\tau) - \Omega_\tau(x,\tau) \tag{6.2a}$$

$$\sigma_\Omega(x) = \begin{pmatrix} s_X\,\Omega_x(x,\tau) \\ 0 \end{pmatrix}. \tag{6.2b}$$

We are now set to apply our truncated series representation to the Markovian version of (6.1). Let $\zeta=1$ so that $\Omega(x,0)=0$. For example, with $N=1$ and $x_0=\bar{x}$, we have

$$\delta_0^{1\prime}(\tau) = \bar{x} - \beta + \frac{\beta\left(1-e^{(1-\eta)\,\delta_0^1(\tau)}\right)}{1-\eta} + (1-\gamma)\,\frac{1}{2}\left(\left(s_1 + s_X\,\delta_1^1(\tau)\right)^2 + s_2^2\right) \tag{6.3a}$$

$$\delta_1^{1\prime}(\tau) = 1 - \left(\kappa + \beta\,e^{(1-\eta)\,\delta_0^1(\tau)}\right)\delta_1^1(\tau), \tag{6.3b}$$

subject to $\delta_0^1(0) = \delta_1^1(0) = 0$. For $N = 2$, we must add terms to the right-hand sides of (6.3) (changing δ_i^1 to δ_i^2): $s_X^2\, \delta_2^2(\tau)$ to (6.3a) and $s_X\,(1-\gamma)\,(s_1 + s_X\,\delta_1^2(\tau))\,\delta_2^2(\tau)$ to (6.3b), where

$$\delta_2^{2\prime}(\tau) = -\frac{1}{2}\,\beta\, e^{(1-\eta)\,\delta_0^2(\tau)}\,(1-\eta)\,\delta_1^2(\tau)^2 - \\ \left(2\,\kappa + \beta\, e^{(1-\eta)\,\delta_0^2(\tau)}\right)\,\delta_2^2(\tau) + 2\,(1-\gamma)\,s_X^2\,\delta_2^2(\tau)^2, \quad (6.4)$$

subject to $\delta_2(0) = 0$.

Special cases. Before we proceed to the numerical solution, there are two cases we can analyze analytically. Both cases involve a linearization of (6.3). These cases can be understood in terms of the dividend-denominated asymptotic forward rate, φ_d. Using (4.7) and $\sigma_\pi = (\eta - 1)\,\sigma_\omega$, we can write the dividend-denominated interest rate and price of risk as

$$r_d = \eta\,\beta + (1-\eta)\left\{\widetilde{\mu}_\phi + (1-\gamma)\,\frac{1}{2}\,\|\sigma_\phi\|^2 - (2-\eta-\gamma)\,\frac{1}{2}\,\|\sigma_\omega\|^2\right\} \quad (6.5a)$$

$$\lambda_d = (\gamma - 1)\,\sigma_\phi - (2-\eta-\gamma)\,\sigma_\omega. \quad (6.5b)$$

Note that when $\eta + \gamma = 2$, the terms involving σ_ω drop out of (6.5), leaving an exponential-affine model of the dividend-denominated term structure.

For the first case, consider $\eta = 1$ ($\rho = 0$). In Section 5 we showed that the solution for $\log(\psi)$ is $\Omega(x,\tau) = \delta_0(\tau) + \delta_1(\tau)\,(x - \bar{x})$ in this case. Since $\omega = \psi^{1/\eta}$, the solution for $\log(\omega)$ is identical. Thus we get the true solution with $N = 1$. Moreover, $\varphi_d = r_d = \beta > 0$, so that convergence to an infinite-horizon is guaranteed. Therefore we can characterize the infinite-horizon problem as a system of algebraic equations,

$$0 = \bar{x} - \beta - \beta\,\delta_0(\infty) + (1-\gamma)\,\frac{1}{2}\left((s_1 + s_X\,\delta_1(\infty))^2 + s_2^2\right) \quad (6.6a)$$

$$0 = 1 - (\kappa + \beta)\,\delta_1(\infty), \quad (6.6b)$$

with the unique solution[33]

$$\delta_0(\infty) = \frac{\bar{x} - \beta}{\beta} + \frac{(1-\gamma)}{\beta}\,\frac{1}{2}\left(\left(s_1 + \frac{s_X}{\kappa+\beta}\right)^2 + s_2^2\right) \quad (6.7a)$$

$$\delta_1(\infty) = \frac{1}{\beta + \kappa}. \quad (6.7b)$$

For the second case, let $\varphi_d \leq 0$. In this case there is no infinite-horizon solution: The wealth–consumption ratio, which is the value of a coupon bond, grows without bound, and its inverse, the consumption–wealth ratio, shrinks to zero as the horizon goes to infinity. The term $\beta\, e^{(1-\eta)\,\delta_0^N(\tau)}$ captures the scale of c/k, and so it goes to

[33] See Section 4 for a different derivation of (6.7).

zero, thereby linearizing the system of ODEs. For large τ, (6.3) becomes

$$\delta_0^{1\prime}(\tau) \doteq \bar{x} - \beta + \frac{\beta}{1-\eta} + (1-\gamma)\,\frac{1}{2}\left(\left(s_1 + s_X\,\delta_1^1(\tau)\right)^2 + s_2^2\right) \tag{6.8a}$$

$$\delta_1^{1\prime}(\tau) \doteq 1 - \kappa\,\delta_1^1(\tau). \tag{6.8b}$$

For $N > 1$, all higher-order coefficients are asymptotically zero, so that the model is asymptotically first-order in the region of nonconvergence. Using (6.8b), we can compute $\lim_{\tau\to\infty}\ \delta_1(\tau) = 1/\kappa$. Inserting this into (6.8a) produces

$$\lim_{\tau\to\infty}\ \delta_0{}'(\tau) = \bar{x} - \beta + \frac{\beta}{1-\eta} + (1-\gamma)\,\frac{1}{2}\left((s_1 + s_X/\kappa)^2 + s_2^2\right). \tag{6.9}$$

Because the model is asymptotically first-order in the region where it does not converge, we have $\lim_{\tau\to\infty}\ \sigma_\omega = (s_X/\kappa,\, 0)$. This suggests that even when we cannot determine the boundary of the region of nonconvergence analytically, we can use our solution method to compute it numerically.

Numerical investigation. For a numerical investigation, let

$$\beta = 0.06,\ \kappa = 2.67,\ \bar{x} = 0.065,\ s_X = 0.126,\ s_1 = 0.16,\ \text{and}\ s_2 = 0.04.$$

The parameter values are all measured per annum, and have been chosen to (roughly) match the monthly moments in Campbell (1993). Numerical solutions for various combinations of η and γ are summarized in Tables 3–7. The first column indicates the order of the approximation, N, which runs from 1 to 8. In the second column, $\Omega_\tau^N(x_0,\tau)$ is computed as a measure of whether there has been convergence. In the tables, $x_0 = .065$ and $\tau = 10^5$ years.[34] Numbers less the 10^{-16} in absolute value are reported as zero.

TABLE 3. Model 1, $\eta = 1$, $\gamma = 2$, $\tau = 10^5$, and $x_0 = 0.065$.

N	$\Omega_\tau^N(x_0,\tau)$	$\delta_0^N(\tau)$	$\delta_1^N(\tau)$	$\delta_2^N(\tau)$	$\delta_3^N(\tau)$	$\delta_4^N(\tau)$	$\delta_5^N(\tau)$	$\delta_6^N(\tau)$
1	0	−2.8416(−1)	3.6630(−1)					
2	0	−2.8416(−1)	3.6630(−1)	0				
3	0	−2.8416(−1)	3.6630(−1)	0	0			
4	0	−2.8416(−1)	3.6630(−1)	0	0	0		
5	0	−2.8416(−1)	3.6630(−1)	0	0	0	0	
6	0	−2.8416(−1)	3.6630(−1)	0	0	0	0	0
7	0	−2.8416(−1)	3.6630(−1)	0	0	0	0	0
8	0	−2.8416(−1)	3.6630(−1)	0	0	0	0	0

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

[34] The initial step size taken in solving the ODEs is 10^{-6} years ($\approx$ 31.5 seconds).

Table 4. Model 1, $\eta = 2$, $\gamma = 2$, $\tau = 10^5$, and $x_0 = 0.065$.

N	$\Omega_\tau^N(x_0, \tau)$	$\delta_0^N(\tau)$	$\delta_1^N(\tau)$	$\delta_2^N(\tau)$	$\delta_3^N(\tau)$	$\delta_4^N(\tau)$	$\delta_5^N(\tau)$	$\delta_6^N(\tau)$
1	0	−2.4934(−1)	3.6403(−1)					
2	0	−2.4914(−1)	3.6402(−1)	9.4148(−4)				
3	0	−2.4914(−1)	3.6402(−1)	9.4253(−4)	−7.3255(−5)			
4	0	−2.4914(−1)	3.6402(−1)	9.4261(−4)	−7.3314(−5)	4.6005(−6)		
5	0	−2.4914(−1)	3.6402(−1)	9.4261(−4)	−7.3318(−5)	4.6032(−6)	−2.2577(−7)	
6	0	−2.4914(−1)	3.6402(−1)	9.4261(−4)	−7.3318(−5)	4.6034(−6)	−2.2586(−7)	7.7784(−9)
7	0	−2.4914(−1)	3.6402(−1)	9.4261(−4)	−7.3318(−5)	4.6034(−6)	−2.2587(−7)	7.7793(−9)
8	0	−2.4914(−1)	3.6402(−1)	9.4261(−4)	−7.3318(−5)	4.6034(−6)	−2.2587(−7)	7.7790(−9)

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

Table 5. Model 1, $\eta = 0$, $\gamma = 2$, $\tau = 10^5$, and $x_0 = 0.065$.

N	$\Omega_\tau^N(x_0, \tau)$	$\delta_0^N(\tau)$	$\delta_1^N(\tau)$	$\delta_2^N(\tau)$	$\delta_3^N(\tau)$	$\delta_4^N(\tau)$	$\delta_5^N(\tau)$	$\delta_6^N(\tau)$
1	0	−3.3570(−1)	3.6861(−1)					
2	0	−3.3591(−1)	3.6862(−1)	−5.4124(−4)				
3	0	−3.3591(−1)	3.6862(−1)	−5.4061(−4)	−4.3393(−5)			
4	0	−3.3591(−1)	3.6862(−1)	−5.4066(−4)	−4.3356(−5)	−2.8664(−6)		
5	0	−3.3591(−1)	3.6862(−1)	−5.4066(−4)	−4.3359(−5)	−2.8645(−6)	−1.5459(−7)	
6	0	−3.3591(−1)	3.6862(−1)	−5.4066(−4)	−4.3359(−5)	−2.8647(−6)	−1.5451(−7)	−6.6105(−9)
7	0	−3.3591(−1)	3.6862(−1)	−5.4066(−4)	−4.3359(−5)	−2.8647(−6)	−1.5451(−7)	−6.6082(−9)
8	0	−3.3591(−1)	3.6862(−1)	−5.4066(−4)	−4.3359(−5)	−2.8647(−6)	−1.5451(−7)	−6.6082(−9)

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

Table 6. Model 1, $\eta = 2$, $\gamma = 1$, $\tau = 10^5$, and $x_0 = 0.065$.

N	$\Omega_\tau^N(x_0, \tau)$	$\delta_0^N(\tau)$	$\delta_1^N(\tau)$	$\delta_2^N(\tau)$	$\delta_3^N(\tau)$	$\delta_4^N(\tau)$	$\delta_5^N(\tau)$	$\delta_6^N(\tau)$
1	0	8.7011(−2)	3.6697(−1)					
2	0	8.7210(−2)	3.6697(−1)	6.8632(−4)				
3	0	8.7210(−2)	3.6697(−1)	6.8632(−4)	−5.4442(−5)			
4	0	8.7210(−2)	3.6697(−1)	6.8638(−4)	−5.4442(−5)	3.5329(−6)		
5	0	8.7210(−2)	3.6697(−1)	6.8638(−4)	−5.4446(−5)	3.5329(−6)	−1.8447(−7)	
6	0	8.7210(−2)	3.6697(−1)	6.8638(−4)	−5.4446(−5)	3.5331(−6)	−1.8447(−7)	7.3644(−9)
7	0	8.7210(−2)	3.6697(−1)	6.8638(−4)	−5.4446(−5)	3.5331(−6)	−1.8447(−7)	7.3644(−9)
8	0	8.7210(−2)	3.6697(−1)	6.8638(−4)	−5.4446(−5)	3.5331(−6)	−1.8447(−7)	7.3643(−9)

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

TABLE 7. Model 1, $\eta = 4$, $\gamma = 0$, $\tau = 10^5$, and $x_0 = 0.065$.

N	$\Omega_\tau^N(x_0, \tau)$	$\delta_0^N(\tau)$	$\delta_1^N(\tau)$	$\delta_2^N(\tau)$	$\delta_3^N(\tau)$	$\delta_4^N(\tau)$	$\delta_5^N(\tau)$	$\delta_6^N(\tau)$
1	7.2641(−3)	7.2684(2)	3.7453(−1)					
2	7.2641(−3)	7.2684(2)	3.7453(−1)	0				
3	7.2641(−3)	7.2684(2)	3.7453(−1)	0	0			
4	7.2641(−3)	7.2684(2)	3.7453(−1)	0	0	0		
5	7.2641(−3)	7.2684(2)	3.7453(−1)	0	0	0	0	
6	7.2641(−3)	7.2684(2)	3.7453(−1)	0	0	0	0	0
7	7.2641(−3)	7.2684(2)	3.7453(−1)	0	0	0	0	0
8	7.2641(−3)	7.2684(2)	3.7453(−1)	0	0	0	0	0

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

TABLE 8. Model 2, $\eta = 1$, $\gamma = 2$, $\tau = 10^5$, and $x_0 = 0.065$.

N	$\Omega_\tau^N(x_0, \tau)$	$\delta_0^N(\tau)$	$\delta_1^N(\tau)$	$\delta_2^N(\tau)$	$\delta_3^N(\tau)$	$\delta_4^N(\tau)$	$\delta_5^N(\tau)$	$\delta_6^N(\tau)$
1	0	−2.3860(−1)	2.5745(−1)					
2	−2.6097(−16)	−2.3860(−1)	2.5745(−1)	5.5093(−11)				
3	−2.1117(−15)	−2.3860(−1)	2.5745(−1)	−4.4649(−11)	−3.1826(−11)			
4	0	−2.3860(−1)	2.5745(−1)	2.5589(−11)	2.0673(−10)	3.2711(−10)		
5	−1.3657(−16)	−2.3860(−1)	2.5745(−1)	1.6848(−12)	−1.2576(−11)	−1.1281(−10)	−2.1945(−9)	
6	0	−2.3860(−1)	2.5745(−1)	−2.8775(−12)	−8.2670(−12)	−8.6877(−11)	−6.0362(−10)	−7.4186(−10)
7	0	−2.3860(−1)	2.5745(−1)	9.6853(−12)	5.4134(−11)	2.5007(−10)	9.6076(−10)	3.4567(−9)
8	−5.4286(−16)	−2.3860(−1)	2.5745(−1)	−1.4021(−11)	−9.6002(−12)	−3.4361(−12)	2.2344(−13)	1.7694(−12)

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

TABLE 9. Model 2, $\eta = 2$, $\gamma = 2$, $\tau = 10^5$, and $x_0 = 0.065$.

N	$\Omega_\tau^N(x_0, \tau)$	$\delta_0^N(\tau)$	$\delta_1^N(\tau)$	$\delta_2^N(\tau)$	$\delta_3^N(\tau)$	$\delta_4^N(\tau)$	$\delta_5^N(\tau)$	$\delta_6^N(\tau)$
1	0	−2.1360(−1)	2.5627(−1)					
2	0	−2.1352(−1)	2.5629(−1)	3.9605(−4)				
3	0	−2.1352(−1)	2.5629(−1)	3.9371(−4)	−2.1841(−5)			
4	0	−2.1352(−1)	2.5629(−1)	3.9374(−4)	−2.1696(−5)	9.7983(−7)		
5	0	−2.1352(−1)	2.5629(−1)	3.9374(−4)	−2.1697(−5)	9.7321(−7)	−3.4951(−8)	
6	0	−2.1352(−1)	2.5629(−1)	3.9374(−4)	−2.1697(−5)	9.7326(−7)	−3.4739(−8)	9.2272(−10)
7	0	−2.1352(−1)	2.5629(−1)	3.9374(−4)	−2.1697(−5)	9.7326(−7)	−3.4740(−8)	9.1939(−10)
8	0	−2.1352(−1)	2.5629(−1)	3.9374(−4)	−2.1697(−5)	9.7326(−7)	−3.4740(−8)	9.1935(−10)

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

Table 10. Model 2, $\eta = 0$, $\gamma = 2$, $\tau = 10^5$, and $x_0 = 0.065$.

N	$\Omega_\tau^N(x_0,\tau)$	$\delta_0^N(\tau)$	$\delta_1^N(\tau)$	$\delta_2^N(\tau)$	$\delta_3^N(\tau)$	$\delta_4^N(\tau)$	$\delta_5^N(\tau)$	$\delta_6^N(\tau)$
1	0	−2.7324(−1)	2.5864(−1)					
2	0	−2.7331(−1)	2.5863(−1)	−2.4897(−4)				
3	0	−2.7332(−1)	2.5863(−1)	−2.5048(−4)	−1.4025(−5)			
4	0	−2.7332(−1)	2.5863(−1)	−2.5050(−4)	−1.4123(−5)	−6.5206(−7)		
5	0	−2.7332(−1)	2.5863(−1)	−2.5050(−4)	−1.4124(−5)	−6.5677(−7)	−2.4830(−8)	
6	0	−2.7332(−1)	2.5863(−1)	−2.5050(−4)	−1.4124(−5)	−6.5682(−7)	−2.5004(−8)	−7.5546(−10)
7	0	−2.7332(−1)	2.5863(−1)	−2.5050(−4)	−1.4124(−5)	−6.5682(−7)	−2.5006(−8)	−7.5994(−10)
8	0	−2.7332(−1)	2.5863(−1)	−2.5050(−4)	−1.4124(−5)	−6.5682(−7)	−2.5006(−8)	−7.5996(−10)

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

Table 11. Model 2, $\eta = 2$, $\gamma = 1$, $\tau = 10^5$, and $x_0 = 0.065$.

N	$\Omega_\tau^N(x_0,\tau)$	$\delta_0^N(\tau)$	$\delta_1^N(\tau)$	$\delta_2^N(\tau)$	$\delta_3^N(\tau)$	$\delta_4^N(\tau)$	$\delta_5^N(\tau)$	$\delta_6^N(\tau)$
1	0	8.7011(−2)	3.6697(−1)					
2	0	8.7210(−2)	3.6704(−1)	6.8655(−4)				
3	0	8.7207(−2)	3.6703(−1)	6.7914(−4)	−5.4488(−5)			
4	0	8.7207(−2)	3.6703(−1)	6.7929(−4)	−5.3845(−5)	3.5389(−6)		
5	0	8.7207(−2)	3.6703(−1)	6.7929(−4)	−5.3856(−5)	3.4968(−6)	−1.8509(−7)	
6	0	8.7207(−2)	3.6703(−1)	6.7929(−4)	−5.3856(−5)	3.4974(−6)	−1.8306(−7)	7.4163(−9)
7	0	8.7207(−2)	3.6703(−1)	6.7929(−4)	−5.3856(−5)	3.4974(−6)	−1.8308(−7)	7.3585(−9)
8	0	8.7207(−2)	3.6703(−1)	6.7929(−4)	−5.3856(−5)	3.4974(−6)	−1.8308(−7)	7.3581(−9)

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

Table 12. Model 2, $\eta = 4$, $\gamma = 0$, $\tau = 10^5$, and $x_0 = 0.065$.

N	$\Omega_\tau^N(x_0,\tau)$	$\delta_0^N(\tau)$	$\delta_1^N(\tau)$	$\delta_2^N(\tau)$	$\delta_3^N(\tau)$	$\delta_4^N(\tau)$	$\delta_5^N(\tau)$	$\delta_6^N(\tau)$
1	1.1423(−2)	1.1426(3)	5.2678(−1)					
2	1.1423(−2)	1.1426(3)	5.2678(−1)	0				
3	1.1423(−2)	1.1426(3)	5.2678(−1)	0	0			
4	1.1423(−2)	1.1426(3)	5.2678(−1)	0	0	0		
5	1.1423(−2)	1.1426(3)	5.2678(−1)	0	0	0	0	
6	1.1423(−2)	1.1426(3)	5.2678(−1)	0	0	0	0	0
7	1.1423(−2)	1.1426(3)	5.2678(−1)	0	0	0	0	0
8	1.1423(−2)	1.1426(3)	5.2678(−1)	0	0	0	0	0

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

TABLE 13. Model 3, $\eta = 1$, $\gamma = 2$, $\tau = 10^5$, and $x_0 = 0.065$.

N	$\Omega_\tau^N(x_0,\tau)$	$\delta_0^N(\tau)$	$\delta_1^N(\tau)$	$\delta_2^N(\tau)$	$\delta_3^N(\tau)$	$\delta_4^N(\tau)$	$\delta_5^N(\tau)$	$\delta_6^N(\tau)$
1	0	−2.1191(−1)	1.9171(−1)					
2	0	−3.5652(−1)	1.9992(−1)	−5.3453(−1)				
3	0	−3.5652(−1)	1.9992(−1)	−5.3453(−1)	0			
4	0	−3.5652(−1)	1.9992(−1)	−5.3453(−1)	0	0		
5	0	−3.5652(−1)	1.9992(−1)	−5.3453(−1)	0	0	0	
6	0	−3.5652(−1)	1.9992(−1)	−5.3453(−1)	0	0	0	0
7	0	−3.5652(−1)	1.9992(−1)	−5.3453(−1)	0	0	0	0
8	0	−3.5652(−1)	1.9992(−1)	−5.3453(−1)	0	0	0	0

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

TABLE 14. Model 3, $\eta = 2$, $\gamma = 2$, $\tau = 10^5$, and $x_0 = 0.065$.

N	$\Omega_\tau^N(x_0,\tau)$	$\delta_0^N(\tau)$	$\delta_1^N(\tau)$	$\delta_2^N(\tau)$	$\delta_3^N(\tau)$	$\delta_4^N(\tau)$	$\delta_5^N(\tau)$	$\delta_6^N(\tau)$
1	0	−1.9194(−1)	1.9092(−1)					
2	0	−3.0409(−1)	1.9849(−1)	−5.3236(−1)				
3	0	−3.0409(−1)	1.9847(−1)	−5.3235(−1)	−9.6958(−4)			
4	0	−3.0408(−1)	1.9847(−1)	−5.3233(−1)	−9.8032(−4)	1.0361(−3)		
5	0	−3.0408(−1)	1.9847(−1)	−5.3233(−1)	−9.8304(−4)	1.0375(−3)	−1.5321(−4)	
6	0	−3.0408(−1)	1.9847(−1)	−5.3233(−1)	−9.8309(−4)	1.0400(−3)	−1.5437(−4)	1.2410(−4)
7	0	−3.0408(−1)	1.9847(−1)	−5.3233(−1)	−9.8310(−4)	1.0400(−3)	−1.5479(−4)	1.2427(−4)
8	0	−3.0408(−1)	1.9847(−1)	−5.3233(−1)	−9.8310(−4)	1.0400(−3)	−1.5480(−4)	1.2457(−4)

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

TABLE 15. Model 3, $\eta = 0$, $\gamma = 2$, $\tau = 10^5$, and $x_0 = 0.065$.

N	$\Omega_\tau^N(x_0,\tau)$	$\delta_0^N(\tau)$	$\delta_1^N(\tau)$	$\delta_2^N(\tau)$	$\delta_3^N(\tau)$	$\delta_4^N(\tau)$	$\delta_5^N(\tau)$	$\delta_6^N(\tau)$
1	0	−2.3854(−1)	1.9252(−1)					
2	0	−4.4260(−1)	2.0138(−1)	−5.3658(−1)				
3	0	−4.4261(−1)	2.0139(−1)	−5.3658(−1)	4.6062(−4)			
4	0	−4.4261(−1)	2.0139(−1)	−5.3659(−1)	4.6515(−4)	−4.3188(−4)		
5	0	−4.4261(−1)	2.0139(−1)	−5.3659(−1)	4.6385(−4)	−4.3117(−4)	−7.2489(−5)	
6	0	−4.4261(−1)	2.0139(−1)	−5.3659(−1)	4.6383(−4)	−4.3018(−4)	−7.2953(−5)	4.9123(−5)
7	0	−4.4261(−1)	2.0139(−1)	−5.3659(−1)	4.6384(−4)	−4.3018(−4)	−7.2751(−5)	4.9041(−5)
8	0	−4.4261(−1)	2.0139(−1)	−5.3659(−1)	4.6384(−4)	−4.3019(−4)	−7.2749(−5)	4.8924(−5)

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

TABLE 16. Model 3, $\eta = 2$, $\gamma = 1$, $\tau = 10^5$, and $x_0 = 0.065$.

N	$\Omega_\tau^N(x_0, \tau)$	$\delta_0^N(\tau)$	$\delta_1^N(\tau)$	$\delta_2^N(\tau)$	$\delta_3^N(\tau)$	$\delta_4^N(\tau)$	$\delta_5^N(\tau)$	$\delta_6^N(\tau)$
1	0	8.7780(−2)	3.6746(−1)					
2	0	8.8514(−2)	3.6746(−1)	2.5409(−3)				
3	0	8.8514(−2)	3.6746(−1)	2.5408(−3)	−4.9953(−5)			
4	0	8.8514(−2)	3.6746(−1)	2.5409(−3)	−4.9953(−5)	2.9314(−6)		
5	0	8.8514(−2)	3.6746(−1)	2.5409(−3)	−4.9955(−5)	2.9314(−6)	−1.2979(−7)	
6	0	8.8514(−2)	3.6746(−1)	2.5409(−3)	−4.9955(−5)	2.9314(−6)	−1.2979(−7)	3.7607(−9)
7	0	8.8514(−2)	3.6746(−1)	2.5409(−3)	−4.9955(−5)	2.9314(−6)	−1.2979(−7)	3.7607(−9)
8	0	8.8514(−2)	3.6746(−1)	2.5409(−3)	−4.9955(−5)	2.9314(−6)	−1.2979(−7)	3.7605(−9)

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

TABLE 17. Model 3, $\eta = 4$, $\gamma = 0$, $\tau = 10^5$, and $x_0 = 0.065$.

N	$\Omega_\tau^N(x_0, \tau)$	$\delta_0^N(\tau)$	$\delta_1^N(\tau)$	$\delta_2^N(\tau)$	$\delta_3^N(\tau)$	$\delta_4^N(\tau)$	$\delta_5^N(\tau)$	$\delta_6^N(\tau)$
1	1.3644(−2)	1.3647(3)	6.0163(−1)					
2	2.5078(−2)	2.5080(3)	6.1910(−1)	6.8732(−1)				
3	2.5078(−2)	2.5080(3)	6.1910(−1)	6.8732(−1)	0			
4	2.5078(−2)	2.5080(3)	6.1910(−1)	6.8732(−1)	0	0		
5	2.5078(−2)	2.5080(3)	6.1910(−1)	6.8732(−1)	0	0	0	
6	2.5078(−2)	2.5080(3)	6.1910(−1)	6.8732(−1)	0	0	0	0
7	2.5078(−2)	2.5080(3)	6.1910(−1)	6.8732(−1)	0	0	0	0
8	2.5078(−2)	2.5080(3)	6.1910(−1)	6.8732(−1)	0	0	0	0

0 signifies less than 10^{-16} in absolute value; $n(d) := n \times 10^d$.

Table 3 presents the results for $\eta = 1$ and $\gamma = 2$. The results in the table confirm our analysis in this case. The value of the time derivative in column 2 indicates that an infinite-horizon solution does indeed exist. The first-order solution appears to be exact, since higher-order terms contribute nothing. Even absent analytical proof, we can always insert the solution into the PDEs, and evaluate the absolute value of the residual as a function of the state variable at the horizon in question. For $\eta = 1$, the residual is indistinguishable from zero for all values of γ and at all horizons.

In Table 4 we move away from the analytically available solutions to $\eta = 2$ and $\gamma = 2$. In this case the solution has infinite order. Here we see that the coefficients converge quite rapidly of a function of N: δ^N_{N-2} has converged to five significant digits. Also note that the coefficients die off rapidly as a function of order. The same observations hold for the coefficients in Tables 5 and 6. Note that in Tables 4–6, δ^N_1 remains close to $1/(\beta + \kappa)$.

Results for Models 2 and 3 for the same preference parameters are shown Tables 8–11 and Tables 13–16, respectively. The models are calibrated so that the unconditional drift and instantaneous variance of $\log(\phi)$ are the same as for Model 1. In terms of convergence, the are qualitatively similar to Model 1, although Model 2 suffers from some rounding error at $\eta = 1$ that is not present in either of the other two models. Quantitatively, we see all three models are essentially the same for $\gamma = 1$.

TABLE 18. Maximum absolute errors for $x \in (-0.12, 0.25)$ for $\eta = 2$, $\gamma = 2$, and $\tau = 10^5$.

N	Model 1	Model 2	Model 3
1	1.7858(−4)	8.4821(−5)	1.0978(−1)
2	3.8123(−6)	1.2870(−6)	7.0186(−5)
3	5.8662(−8)	1.4172(−8)	1.5077(−5)
4	6.6224(−10)	1.1647(−10)	5.8699(−7)
5	5.0328(−12)	6.8003(−13)	9.2199(−8)
6	1.2101(−14)	1.9513(−15)	3.3115(−9)
7	3.8858(−16)	0	4.1261(−10)
8	0	0	1.3604(−11)

0 signifies less than 10^{-16} in absolute value;

$n(d) := n \times 10^d$.

There remains the question as to how well $\Omega^N(x, \infty)$ fits the defining restriction. Since the solution method is local in nature, the fit will be perfect at x_0 and decay as we move away, although the decay need not be monotonic in $|x - x_0|$. There are two related issues here. First, what is the range over which we desire a good fit, and, second, how good a fit should the fit be? To help determine the appropriate range, we can compute the unconditional distribution of the state variable. In our example for Model 1, $x \sim \mathcal{N}\left(\bar{x}, s_X/\sqrt{2\,\kappa}\right) = \mathcal{N}\,(0.065, 0.054526)$. A range centered on $\bar{x}$ that includes more than 99.9 percent of the PDF is $(-0.12, 0.25)$. Table 18 shows the maximum absolute error over this region for $(\eta, \gamma) = (2, 2)$. Also shown are the maximum absolute errors over the same region for Models 2 and 3. Even the first-order approximation for Models 1 and 2 and the second-order approximation for Model 3 are reasonably accurate over the region.

Regions of nonconvergence. For Model 1, we can easily compute the boundary of the regions of nonconvergence using (5.17):

$$\varphi_d = \eta\,\beta + (1-\eta)\,(0.065 + (1-\gamma)\,0.0223)\,. \tag{6.10}$$

With $\beta = 0.06$, $\eta = 4$, and $\gamma = 0$, we have $\varphi_d = -0.0291$. Table 7 shows results for $\eta = 4$ and $\gamma = 0$. The time derivative is clearly not zero, even at a horizon of 10^5 years. Moreover, all of the coefficients higher than first-order are effectively zero at this horizon. The residual from the PDE is indistinguishable from zero at this horizon. Also note that $\delta_1^N = 1/\kappa = 0.37453$ as expected.

Using (6.10), we can map out the regions of nonconvergence. Panel (a) of Figure 2 shades the regions where $\varphi_d \leq 0$ using $\beta = 0.06$. Standard preferences are plotted as the rectangular hyperbola $\eta\,\gamma = 1$. Note that there is no infinite-horizon solution for standard preferences unless $0.26 < \gamma < 4.65$. This rules out the level of risk-aversion that has previously been found consistent with the moments of asset returns and consumption growth. Panel (b) of Figure 2 illustrates the effect of lowering the rate of time preference to $\beta = 0.02$ on the regions of nonconvergence. We see that a sizable fraction of the region Campbell studied is nonconvergent in this case. In the limit as $\beta \to 0$, the regions of nonconvergence form a checkerboard, approaching the point $(\eta, \gamma) = (1, 3.9195)$. Figures 3 and 4 show similar results for Models 2 and 3, respectively.

Two state variables. Now consider a model with two state variables. Augment the previous model with stochastic volatility. In particular, let

$$\begin{aligned} d\log(\phi) &= x\,dt + s_1\,dW_1 + \sqrt{y}\,dW_2 \\ dx &= \kappa\,(\bar{x} - x)\,dt + s_X\,dW_1 \\ dy &= \kappa_Y\,(\bar{y} - y)\,dt + s_Y\,\sqrt{y}\,dW_2. \end{aligned}$$

We keep the values for s_1, $\bar{x}$, κ, and s_X from the previous example, and let $\bar{y} = 0.04^2$ so that $\sqrt{\bar{y}} = .04$ ($= s_2$ from the previous example). Finally let $\kappa_Y = 1$ and $s_Y = .02$.[35] Table 19 shows some results for $\eta = 2$ and $\gamma = 2$. There are $(N+1)\,(N+2)/2$ coefficients $\delta_{ij}(\tau)$ for which $i + j \leq N$. The upper-left number in each block is the constant term $\delta_{00}(\tau)$. The remaining numbers in the first row of each block are the coefficients on powers of $x - \bar{x}$, while the remaining numbers in the first column

[35] These values have been chosen arbitrarily.

FIGURE 2. Areas of nonconvergence for Model 1 are shaded. Panel (a) uses $\beta = 0.06$, while panel (b) uses $\beta = 0.02$.

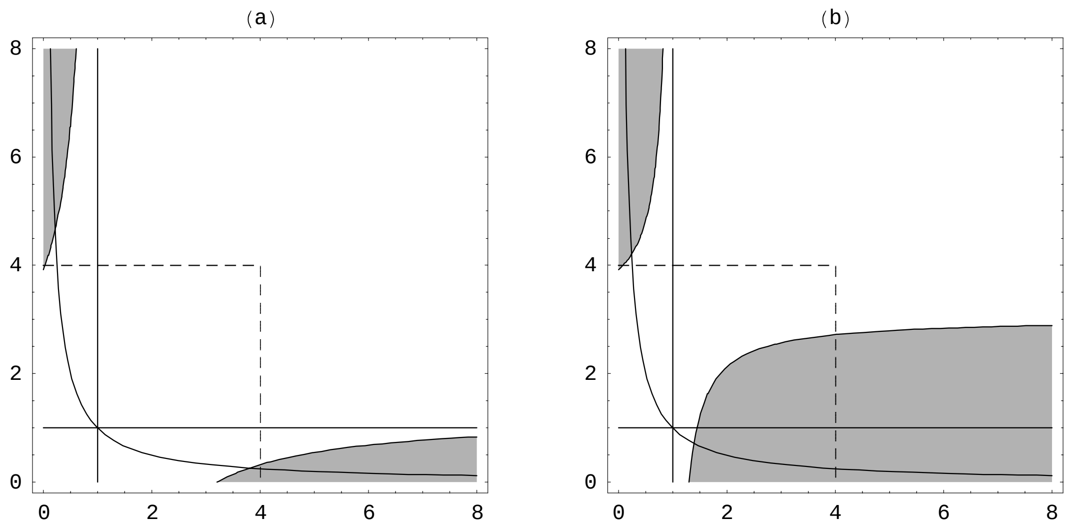


FIGURE 3. Areas of nonconvergence for Model 2 are shaded. Panel (a) uses $\beta = 0.06$, while panel (b) uses $\beta = 0.02$.

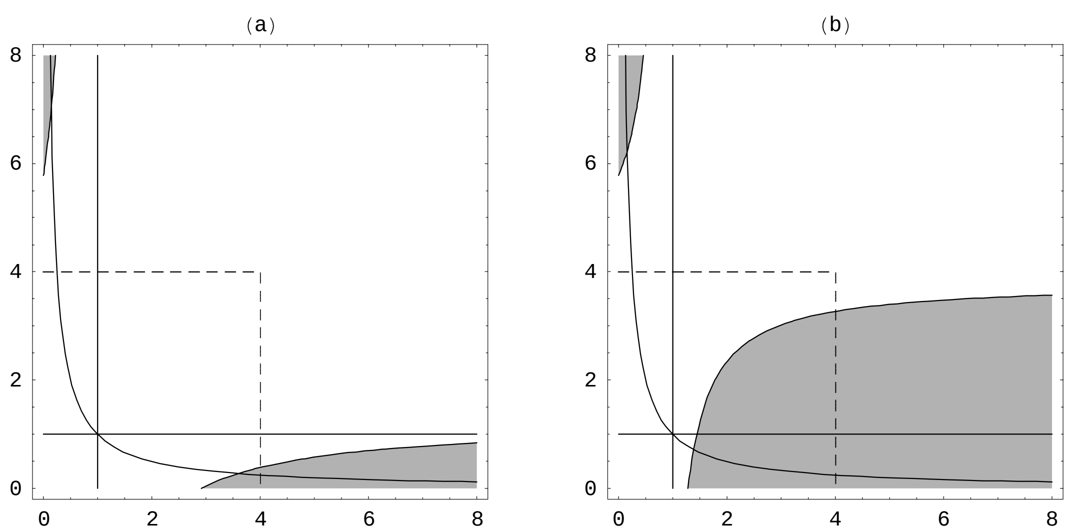
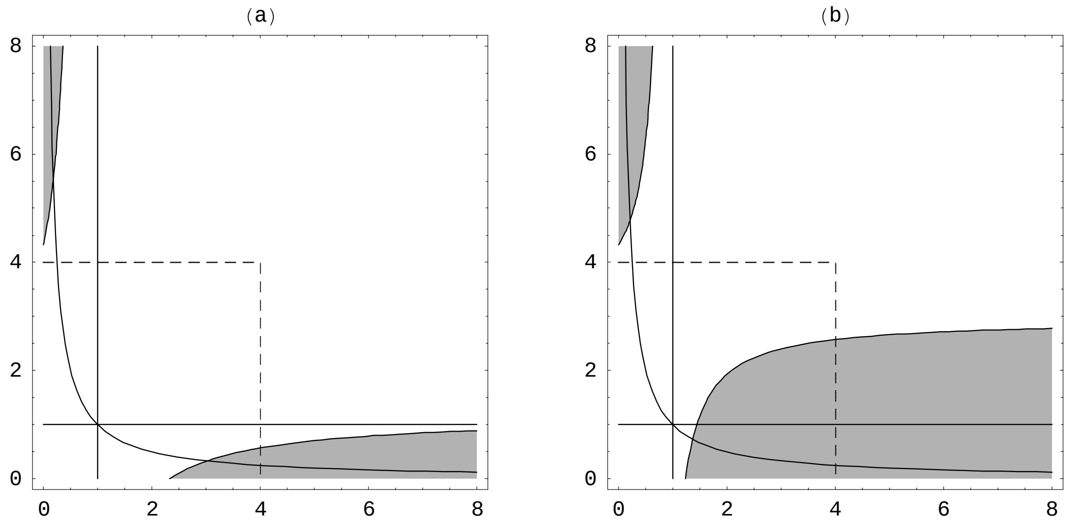


FIGURE 4. Areas of nonconvergence for Model 3 are shaded. Panel (a) uses $\beta = 0.06$, while panel (b) uses $\beta = 0.02$.



of each block are the coefficients on powers of $y - \bar{y}$. The time derivatives are essentially zero, indicating the existence of an infinite-horizon solution for these parameter values. As in the one-factor example, the coefficients converge rapidly as a function of N. Note that the coefficients for x are little changed from the previous example, while at the same time y enters the solution with an impact of the same magnitude as x. In Table 20, the coefficient of relative risk aversion is set to one, which is the CAPM. In this case, the model reverts to a one-factor model: The volatility of the return on the market plays no role. In Table 21, the parameters are $\eta = 5$ and $\gamma = 5$. In this case, the coefficients for y do not decay as rapidly as previously with respect to the order.

Appendix A. The absence of arbitrage

We assume the existence of a *state-price deflator*, which follows a strictly positive Itô process $m(t)$ that we write as:

$$\frac{dm(t)}{m(t)} = -r(t)\,dt - \lambda(t)^\top dW(t)\,, \tag{A.1}$$

where "$^\top$" denotes the transpose, $r(t)$ is the instantaneous rate of interest and $\lambda(t)$ is the price of risk. Observe that we are free to model $r(t)$ and $\lambda(t)$ independently, as long as a solution to (A.1) exists and

$$\exp\left(\int_{s=0}^{t} -\frac{1}{2}\|\lambda(s)\|^2\,ds + \lambda(s)^\top dW(s)\right)$$

TABLE 19. $\eta = 2$, $\gamma = 2$, $\tau = 10^5$, $x_0 = 0.065$, $y_0 = 0.0016$.

			x				
N	Ω_τ^N	y	0	1	2	3	4
1	0	0	−2.4915(−1)	3.6404(−1)			
		1	−4.5584(−1)				
2	0	0	−2.4896(−1)	3.6402(−1)	9.4132(−4)		
		1	−4.5576(−1)	−3.3897(−3)			
		2	3.7764(−3)				
3	0	0	−2.4895(−1)	3.6402(−1)	9.4236(−4)	−7.3243(−5)	
		1	−4.5576(−1)	−3.3944(−3)	3.4113(−4)		
		2	3.7833(−3)	−5.6083(−4)			
		3	3.4483(−4)				
4	0	0	−2.4895(−1)	3.6402(−1)	9.4244(−4)	−7.3302(−5)	4.5999(−6)
		1	−4.5576(−1)	−3.3947(−3)	3.4145(−4)	−2.6436(−5)	
		2	3.7838(−3)	−5.6147(−4)	5.7989(−5)		
		3	3.4532(−4)	−5.8242(−5)			
		4	2.3228(−5)				

0 signifies less than 10^{-16} in absolute value. $n(d) := n \times 10^d$.

is a martingale.[36]

A state-price deflator $m(t)$ guarantees that asset prices are free of arbitrage possibilities. The price of any asset (expressed in a given unit of account) is determined by the formula that its *deflated gain* is a martingale. To see what this means, consider an asset with *cumulative dividend* $D(t)$ and value $S(t)$, both Itô processes. For simplicity of exposition, assume that $D(t)$ is locally riskless. Let the dynamics of S and D be given by

$$dS(t) = \bar{\mu}_S(t)\,dt + \bar{\sigma}_S(t)^\top dW(t), \quad \text{and} \quad dD(t) = Z(t)\,dt,$$

where $Z(t)$ is the flow of dividends. The *gain* is the sum of the asset's value and its cumulative dividend, $G(t) := S(t) + D(t)$, while the deflated gain is $G(t)\,m(t)$. To say that $G(t)\,m(t)$ is a martingale is equivalent to saying that the price process $S(t)$ obeys

$$S(t) = E_t\left[\int_{s=t}^{T}\left(\frac{m(s)}{m(t)}\right) Z(s)\,ds + \left(\frac{m(T)}{m(t)}\right) S(T)\right], \tag{A.2}$$

[36]This condition is referred to as an integrability condition. The example given in Cox, Ingersoll, Jr., and Ross (1985b) fails this condition.

TABLE 20. $\eta = 2$, $\gamma = 1$, $\tau = 10^5$, $x_0 = 0.065$, $y_0 = 0.0016$.

			x				
N	Ω_τ^N	y	0	1	2	3	4
1	0	0	8.7011(−2)	3.6697(−1)			
		1	0				
2	0	0	8.7210(−2)	3.6697(−1)	6.8632(−4)		
		1	0	0			
		2	0				
3	0	0	8.7210(−2)	3.6697(−1)	6.8632(−4)	−5.4442(−5)	
		1	0	0	0		
		2	0	0			
		3	0				
4	0	0	8.7210(−2)	3.6697(−1)	6.8638(−4)	−5.4442(−5)	3.5329(−6)
		1	0	0	0	0	
		2	0	0	0		
		3	0	0			
		4	0				

0 signifies less than 10^{-16} in absolute value. $n(d) := n \times 10^d$.

for any $T > t$, where E_t stands for the expectation conditional on time-t information. A direct implication of the pricing equation (A.2) is the no-arbitrage condition:

$$\bar{\mu}_S(t) + Z(t) = r(t)\, S(t) + \lambda(t)^\top \bar{\sigma}_S(t). \tag{A.3}$$

Fixed income assets play a central role in the body of the paper. Let $p(t,T)$ denote the price at time t of a zero-coupon bond paying one unit of account at time T. According to the pricing formula (A.2), the terminal condition $p(T,T) = 1$ implies

$$p(t,T) = E_t\left[\frac{m(T)}{m(t)}\right],$$

so that the term structure theory reduces to the problem of producing conditional forecasts of the state-price deflator. The value of a coupon bond with a face value of θ that pays one unit continuously until it matures at time T is given by

$$\varpi(t) = E_t\left[\int_{s=t}^{T} \frac{m(s)}{m(t)}\, ds + \theta\, \frac{m(T)}{m(t)}\right] = \int_{s=t}^{T} p(t,s)\, ds + \theta\, p(t,T).$$

TABLE 21. $\eta = 5$, $\gamma = 5$, $\tau = 10^5$, $x_0 = 0.065$, $y_0 = 0.0016$.

N	Ω_τ^N	y	x: 0	1	2	3	4
1	0	0	−4.5912(−1)	3.2825(−1)			
		1	−1.3742(0)				
2	0	0	−4.5832(−1)	3.2745(−1)	1.4076(−2)		
		1	−1.3640(0)	−1.6253(−1)			
		2	5.5135(−1)				
3	0	0	−4.5829(−1)	3.2739(−1)	1.4247(−2)	−3.3503(−3)	
		1	−1.3633(0)	−1.6479(−1)	4.8315(−2)		
		2	5.6024(−1)	−2.3310(−1)			
		3	3.8396(−1)				
4	0	0	−4.5829(−1)	3.2739(−1)	1.4256(−2)	−3.3732(−3)	4.8814(−4)
		1	−1.3633(0)	−1.6489(−1)	4.8649(−2)	−7.6931(−3)	
		2	5.6054(−1)	−2.3458(−1)	3.9485(−2)		
		3	3.8537(−1)	−5.7982(−2)			
		4	−4.8659(−2)				

0 signifies less than 10^{-16} in absolute value. $n(d) := n \times 10^d$.

Now consider the asset whose value is given in (A.2). Assume the dividend is strictly positive and $S(T) = \theta\, Z(T)$. Then we can divide both sides of (A.2) by $Z(t)$:

$$\begin{aligned} \frac{S(t)}{Z(t)} &= E_t\left[\int_{s=t}^{T} \left(\frac{m(s)\,Z(s)}{m(t)\,Z(t)}\right) ds + \left(\frac{m(T)\,Z(T)}{m(t)\,Z(t)}\right) \frac{S(T)}{Z(T)}\right] \\ &= E_t\left[\int_{s=t}^{T} \frac{m_d(s)}{m_d(t)}\, ds + \theta\, \frac{m_d(T)}{m_d(t)}\right] \\ &= \int_{s=t}^{T} p_d(t,s)\, ds + \theta\, p_d(t,T), \end{aligned}$$

where $m_d(t) := m(t)\,Z(t)$ and $p_d(t,T) := E_t[m_d(T)/m_d(t)]$. We refer to m_d as the dividend-denominated state–price deflator and $p_d(t,T)$ as the dividend-denominated zero-coupon bond. Following (A.1), we can apply Itô's lemma to $m_d(t)$, producing $dm_d(t)/m_d(t) = -r_d(t)\,dt - \lambda_d(t)^\top dW(t)$, where

$$r_d(t) = r(t) - \mu_Z(t) + \lambda(t)^\top \sigma_Z(t) \quad \text{and} \quad \lambda_d(t) = \lambda(t) - \sigma_Z(t),$$

where $dZ(t)/Z(t) = \mu_Z(t)\,dt + \sigma_Z(t)^\top dW(t)$.

Appendix B. Asymptotics and regions of convergence

In this section we examine the relation between the value of an annuity and the value of a zero-coupon bond with the same maturity as the maturity goes to infinity. Particular attention is paid to the case where no infinite-horizon solution exists due to nonpositive asymptotic forward rates. As it turns out, the value of a coupon bond is proportional to the value of a discount bond of the same maturity in the limit as the horizon goes to infinity. As discussed above, if the asymptotic forward rate exists and is positive, then the value of a coupon bond converges to the value of a perpetuity as the horizon goes to infinity. When the asymptotic forward rate is negative, the value of a coupon bond grows without bound. Nevertheless, the relative dynamics of the coupon bond does converge.

Let $p(t,T)$ be the price at time t of a zero-coupon bond that pays one unit at time T. The relative dynamics of bond prices is given by

$$\frac{dp(t,T)}{p(t,T)} = \mu_p(t,T)\,dt + \sigma_p(t,T)^\top dW(t).$$

For this section, we assume the existence of the following two limits:

$$\lim_{T\to\infty} \mu_p(t,T) = \mu_p(t,\infty) \quad \text{and} \quad \lim_{T\to\infty} \sigma_p(t,T) = \sigma_p(t,\infty). \tag{B.1}$$

Define the forward rate as $f(t,T) = -\partial \log(p(t,T))/\partial T$. The dynamics of forward rates can be derived from the dynamics of bond prices:

$$df(t,T) = -\frac{\partial}{\partial T}\, d\log(p(t,T)) = \mu_f(t,T)\,dt + \sigma_f(t,T)^\top dW(t),$$

where

$$\mu_f(t,T) = -\frac{\partial \mu_p(t,T)}{\partial T} + \sigma_p(t,T)^\top \frac{\partial \sigma_p(t,T)}{\partial T} \tag{B.2a}$$

$$\sigma_f(t,T) = -\frac{\partial \sigma_p(t,T)}{\partial T}. \tag{B.2b}$$

Define the asymptotic forward rate as

$$\lim_{T\to\infty} f(t,T) = \varphi \tag{B.3}$$

when the limit exists, where φ is a finite constant. When the asymptotic forward rate exists, $\lim_{T\to\infty} \mu_f(t,T) = 0$ and $\lim_{T\to\infty} \sigma_f(t,T) = 0$. The limits (B.1) are sufficient to guarantee the existence of the asymptotic forward rate.

The value of a coupon bond with a face value of $\theta \geq 0$ that pays a continuous unit coupon is given by $\varpi(t,T) = \int_t^T p(t,s)\,ds + \theta\, p(t,T)$. The relative dynamics of the value of this coupon bond can be written as

$$\frac{d\varpi(t,T)}{\varpi(t,T)} = \mu_\varpi(t,T)\,dt + \sigma_\varpi(t,T)^\top dW(t),$$

where[37]

$$\mu_\varpi(t,T) = \frac{\int_t^T p(t,s)\,\mu_p(t,s)\,ds - 1 + \theta\, p(t,T)\,\mu_p(t,T)}{\varpi(t,T)} \tag{B.4a}$$

$$\sigma_\varpi(t,T) = \frac{\int_t^T p(t,s)\,\sigma_p(t,s)\,ds + \theta\, p(t,T)\,\sigma_p(t,T)}{\varpi(t,T)}. \tag{B.4b}$$

Consider the limiting values for μ_ϖ and σ_ϖ. Note that if both numerators and both denominators in (B.4) diverge, then L'Hopital's rule delivers

$$\lim_{T\to\infty} \mu_\varpi(t,T) = \lim_{T\to\infty} \mu_p(t,T) \quad \text{and} \quad \lim_{T\to\infty} \sigma_\varpi(t,T) = \lim_{T\to\infty} \sigma_p(t,T). \tag{B.5}$$

First note that unless $\lim_{T\to\infty} p(t,T) = 0$, the condition (that both numerators and both denominators diverge) holds. Thus the condition is definitely true if $\varphi < 0$, since bond prices diverge.[38] On the other hand, the condition is definitely false if $\varphi > 0$, since $\lim_{T\to\infty} p(t,T) = 0$ and the ratio test shows that $\lim_{T\to\infty} \varpi(t,T)$ converges as well. If $\varphi = 0$, the condition will be true as long as bond prices do not converge to zero. However, if they do, additional analysis is needed to determine whether the condition holds or not.

An illustration. As an illustration, consider a Markovian setup where bond prices are of the exponential-affine class, so that $p(t,T) = P(X(t), T-t)$, where $P(x,\tau) = \exp(A(\tau) + B(\tau)^\top x)$ and where the dynamics of the Markovian state variables are given by

$$dX(t) = \mu_X(X(t))\,dt + \sigma_X(X(t))^\top dW(t).$$

Applying Ito's lemma to the bond price function we get

$$\begin{aligned}\mu_p(t,t+\tau) &= \mu_X^\top B(\tau) + \frac{1}{2}\operatorname{tr}\left[\sigma_X\,\sigma_X^\top B(\tau)\,B(\tau)^\top\right] - A'(\tau) - B'(\tau)^\top X(t)\\ \sigma_p(t,t+\tau) &= \sigma_X^\top B(\tau).\end{aligned}$$

The forward rate is given by $-A'(\tau) - B'(\tau)^\top X(t)$. If the asymptotic forward rate exists, then

$$\lim_{\tau\to\infty} A'(\tau) = -\varphi \quad \text{and} \quad \lim_{\tau\to\infty} B(\tau) = \mathcal{B},$$

where $\mathcal{B}$ is a vector of finite constants. Therefore, for $\varphi < 0$, the asymptotic dynamics of an annuity are given by

$$\begin{aligned}\lim_{\tau\to\infty} \mu_\varpi(t,t+\tau) &= \mu_X^\top \mathcal{B} + \frac{1}{2}\operatorname{tr}\left[\sigma_X\,\sigma_X^\top \mathcal{B}\,\mathcal{B}^\top\right] + \varphi\\ \lim_{\tau\to\infty} \sigma_\varpi(t,t+\tau) &= \sigma_X^\top \mathcal{B}.\end{aligned}$$

[37] The relative dynamics are well-behaved everywhere except at time T if $\theta = 0$.

[38] Another way to see this is to consider the behavior of $p(t,T)/\varpi(t,T)$ for large T. When $\varphi < 0$, both $p(t,T)$ and $\varpi(t,T)$ diverge, and we can appeal to L'Hopital's rule:

$$\lim_{T\to\infty} \frac{p(t,T)}{\varpi(t,T)} = \lim_{T\to\infty} \frac{\partial p(t,T)/\partial T}{\partial \varpi(t,T)/\partial T} = \lim_{T\to\infty} \frac{\partial p(t,T)/\partial T}{p(t,T)} = -\varphi, \tag{B.6}$$

which shows that when the asymptotic forward rate is negative, the value of an annuity is asymptotically proportional to the value of a very long discount bond.

The risk–return condition for an asymptotic annuity is

$$\hat{\mu}_X^\top \mathcal{B} + \frac{1}{2}\,\mathrm{tr}\left[\sigma_X\,\sigma_X^\top \mathcal{B}\,\mathcal{B}^\top\right] + \varphi = r,$$

where $\hat{\mu}_X = \mu_X - \lambda^\top \sigma_X$ is the risk adjusted drift of X. Given the functions $\hat{\mu}_X$, σ_X, and r, this condition can be solved for $\mathcal{B}$ and φ. For example, let there be a single state variable with O–U dynamics, so that $\hat{\mu}_X = \kappa\,(\bar{X} - x)$, $\sigma_X = s_X$, and $r = x$. Then we have

$$\kappa\,(\bar{X} - x)\,\mathcal{B} + \frac{1}{2}\,s_X^2\,\mathcal{B}^2 + \varphi = x,$$

or, by undetermined coefficients,

$$\mathcal{B} = \frac{-1}{\kappa} \quad \text{and} \quad \varphi = \bar{X} - \frac{1}{2}\left(\frac{s_X}{\kappa}\right)^2. \tag{B.7}$$

To establish the correctness of (B.7) for $\varphi = 0$, eliminate $\bar{X}$ using $\varphi = 0$ as given in (B.7), solve for bond prices, and take the limit. In this case, bond prices converge to a positive value, $\exp\left(-s_X^2/(4\,\kappa^3) - x/\kappa\right)$, thereby establishing the condition that delivers the result for $\varphi = 0$ as well.

The weak form of the expectations hypothesis. Let the forward rate be defined as $f(t,s) := -\partial \log(p(t,s))/\partial s$. The weak form of the expectations hypothesis states that $f(t,s) = E_t[r(s)] + \ell(s-t)$, where $\ell(\tau)$ is a deterministic function of τ and $\ell(0) = 0$.[39] If the interest rate is Gaussian and the price of risk is constant, then the weak form will hold. Let the dynamics of $E_t[r(u)]$ be given $dE_t[r(u)] = \hat{\sigma}_r(t,s)^\top dW(t)$. Then $df(t,s) = -\ell'(s-t)\,dt + \hat{\sigma}_r(t,s)^\top dW(t)$. Applying Itô's lemma to $p(t,s) = \exp\left(-\int_{u=t}^{s} f(t,u)\,du\right)$, produces

$$d\log(p(t,s)) = (r(t) + \ell(s-t))\,dt - \sigma_p(t,u)^\top dW(t),$$

where

$$\sigma_p(t,s) = -\int_{u=t}^{s} \hat{\sigma}_r(t,u)\,du. \tag{B.8}$$

Appendix C. Inhomogeneous terminal reward

When the terminal reward is inhomogeneous, utility $U(\{c\}) = V(0)$ remains homothetic and can be transformed into a homogeneous form. However the process for continuation utility $\{V\}$ cannot be linearly homogeneous in consumption, and therefore cannot be represented as $V(t) = c(t)\,\psi(t)$ for all $t \le T$, except in the limit as $T \to \infty$ (in which case the terminal reward is irrelevant). The problem is that the transformation that makes $U(\{c\})$ homogeneous depends on the horizon. As time advances, the transformation changes. We illustrate these ideas with a simple example.

[39] The strong form of the expectations hypothesis states that $\ell(\tau) \equiv 0$.

Simple example. In this example, the rate of consumption is constant: $c(t) = c_0$ for all $t \le T$. In this case the central restriction can be written as the ODE

$$V'(t) + V(t)\,\beta\,u(c_0/V(t)) = 0, \quad \text{subject to } V(T) = \zeta\,c_0 + \xi,$$

where $V'(t)$ denotes the derivative with respect to time. We have added an inhomogeneous term to the boundary condition, $\xi \ge 0$. The solution is

$$V(t) = \begin{cases} \Big(q(T-t)\,c_0^{\rho} + \big(1 - q(T-t)\big)\,(\zeta\,c_0 + \xi)^{\rho}\Big)^{1/\rho} & \text{for } \rho \neq 0 \\ c_0^{q(T-t)}\,(\zeta\,c_0 + \xi)^{1-q(T-t)} & \text{for } \rho = 0, \end{cases} \tag{C.1}$$

where $q(\tau) = 1 - e^{-\beta\,\tau}$. We can compute the wealth–consumption ratio as we did in the body of the paper (see Footnote 16): $V_c/f_c = \pi$. Therefore

$$\pi(t) = \frac{q(T-t) + (1 - q(T-t))\,\zeta\,c_0^{1-\rho}\,(\zeta\,c_0 + \xi)^{\rho-1}}{\beta},$$

for all values of ρ.

For $\xi = 0$, the boundary is homogeneous, the wealth–consumption ratio is independent of consumption,

$$\pi(t) = \frac{q(T-t) + (1 - q(T-t))\,\zeta^{\rho}}{\beta},$$

and continuation utility is linearly homogeneous in consumption, $V(t) = c_0\,\psi(t)$, where

$$\psi(t) = \begin{cases} \big(q(T-t) + \big(1 - q(T-t)\big)\,\zeta^{\rho}\big)^{1/\rho} & \text{for } \rho \neq 0 \\ \zeta^{1-q(T-t)} & \text{for } \rho = 0. \end{cases}$$

For $\rho = \zeta = 0$, the unique solution is $V(t) = \psi(t) = 0$.

For $\zeta = 0$, continuation utility V is not homogeneous in consumption:

$$V(t) = \Big(q(T-t)\,c_0^{\rho} + \big(1 - q(T-t)\big)\,\xi^{\rho}\Big)^{1/\rho}.$$

Nevertheless, $\pi(t) = q(T-t)/\beta$ is independent of consumption, indicating that utility is homothetic. There is of course a transformation of $V(0)$ that is homogeneous. Define $\hat{V}(t) = \Upsilon^{T}(V(t))$, where

$$\Upsilon^{\tau}(x) = \frac{x^{\rho} - (1 - q(\tau))\,\xi^{\rho}}{q(\tau)},$$

so that $\hat{V}(0) = c_0$. However, since the transformation is horizon-dependent,

$$\hat{V}(t) = \frac{q(T-t)\,c_0^{\rho} + q(t)\,\xi^{\rho}}{q(T)}$$

is not homogeneous for all $t \le T$. For $\rho = 0$,

$$V(t) = c_0^{q(T-t)}\,\xi^{1-q(T-t)},$$

where the wealth–consumption ratio is $\pi(t) = q(T-t)/\beta$. It is the inhomogeneous problem that Duffie and Epstein (1992a) solve.

For the case where $\zeta > 0$ and $\xi > 0$, $\pi(t)$ depends on consumption (unless $\rho = 1$), which implies utility is not homothetic. Indeed, there is no transformation of $V(0)$ that is homogeneous in this case unless $\rho = 1$.

The wealth–consumption ratio more generally. When continuation utility is not linearly homogeneous in consumption, we need to start from scratch. If utility is homothetic, then there is a "solution" for continuation utility in terms of $c(t)$ and an arbitrary second state variable ψ: $V = g(c, \psi)$. Homotheticity (combined with linear investment technology) guarantees that the wealth–consumption ratio is independent of consumption:

$$\frac{g_c(c,\psi)}{f_c(c, g(c,\psi))} = \pi. \tag{C.2}$$

The solution to (C.2) is

$$g(c,\psi) = \begin{cases} \left(c^\rho \, \beta \, \pi + \mathcal{C}(\psi)\right)^{1/\rho} & \text{for } \rho \neq 0 \\ c^{\beta\,\pi}\, \mathcal{C}_0(\psi) & \text{for } \rho = 0, \end{cases} \tag{C.3}$$

where $\mathcal{C}(\psi)$ and $\mathcal{C}_0(\psi)$ are arbitrary functions of ψ.

For homogeneous utility, it is convenient to choose $\mathcal{C}(\psi) = 0$ and $\mathcal{C}_0(\psi) = \psi$. (For $\rho = 0$, $\beta\,\pi = 1$, so continuation utility is indeed homogeneous in this case.) For inhomogeneous utility, we consider two cases. First, for $\rho \neq 0$, we can accommodate the example from above where $\zeta = 0$ and $\xi > 0$ as follows. Let $\mathcal{C}(\psi) = (1 - \beta\,\pi)\,\psi$ and $\mathcal{C}_0(\psi) = \psi^{1-\beta\,\pi}$, where $\beta\,\pi(t) = q(T-t)$.

Second, for the inhomogeneous case when $\rho = 0$, we can choose $\mathcal{C}_0(\psi) = \psi$. The terminal reward imposes the condition $g(c,\psi) = \xi$, which implies $\pi(T) = 0$ and $\psi(T) = \xi$. The following specification is consistent with the restrictions:

$$g(c,\psi) = c^{q(T-t)}\,\psi \quad \text{and} \quad \pi = \frac{q(T-t)}{\beta}. \tag{C.4}$$

The central restriction is given by

$$q\,\widetilde{\mu}_c + \widetilde{\mu}_\psi + \alpha\,\frac{1}{2}\,\|q\,\sigma_c + \sigma_\psi\|^2 - \beta\,\log(\psi) = 0, \quad \text{subject to } \psi(T) = \xi. \tag{C.5}$$

There are three versions of (C.5), the unification of which is parallel to that for the three versions of (2.8). The supporting price system can be computed from

$$\bar{f}_c(c, \Upsilon(c^q\,\psi)) = \beta\,\psi^\alpha\,c^{\alpha\,q-1} \tag{C.6a}$$

$$\bar{f}_v(c, \Upsilon(c^q\,\psi)) = -\beta\left\{1 + \alpha\left((q-1)\,\log(c) + \log(\psi)\right)\right\}. \tag{C.6b}$$

The supporting returns process is given by (3.9a) and $\sigma_\phi = \lambda + \alpha\,(q(T-t)\,\sigma_c + \sigma_\psi)$. The optimal portfolio weights can be obtained by eliminating σ_c from this expression and the price of risk: $\sigma_\phi = (1 - \alpha\,q(T-t))^{-1}\,(\lambda + \alpha\,\sigma_\psi)$. The solution to the resulting linear system is consistent with Theorem 2 in Schroder and Skiadas (1999).

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