MODELING NEGATIVE AUTOREGRESSION IN CONTINUOUS TIME

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Abstract. A continuous-time first-order autoregressive process cannot display negative autoregression at any sampling horizon, but a first-order system of two processes where only one is observed can produce a discretely sampled AR(1) with negative first-order autocorrelation. A parsimonious example is derived.

1. Introduction

Discretely sampled data are sometimes found to fit an AR(1) model with negative first-order autocorrelation. If the underlying data are continuous, then there must by necessity be a richer underlying data-generating process. In this paper I assume the underlying data-generating process is a Markovian system of two continuous-time Ito processes. The observed process, consequently, is not Markovian by itself.

The analysis that follows is closely related to Bergstrom (1983),[1] although he does not treat the issue at hand. Bergstrom's analysis is more general on a number of counts. He treats higher order systems and as well as flow variables. He also discusses estimation, a topic not addressed here. However, Bergstrom's analysis is less general on other counts. Bergstrom explicitly excludes the case studied here, in which a second-order system produces a first-order result. (As Bergstrom notes, his general approach applies to this case.) There is a more important way in which his analysis is more restrictive. For second-order (and higher) stochastic differential equations, he requires that the observable variable be differentiable (in the mean square sense) with respect to time. This has the advantage in the issue at hand of picking out a unique model as we will see below. However, if one treats the observable and unobservable processes symmetrically in a system of Ito processes, Bergstrom's requirement seems arbitrary. Subject matter considerations may prove useful on this count.

In Section 2, I present a continuous-time model that replicates negative first-order autocorrelation. In Section 3, I demonstrate that the proposed model does what is

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The views expressed herein are the author's and do not necessarily reflect those of the Federal Reserve Bank of Atlanta or the Federal Reserve System.

[1]For good introductions to this material, see Bergstrom (1984) and Harvey (1989, chapter 9). The derivation of discrete-time models from continuous-time models in the econometrics literature relies on the notion of continuous-time white noise taken from the engineering literature. The connection between this approach and that of Ito processes can be found in Arnold (1974) and Øksendal (1995).

claimed. In Section 4, I derive the ARMA representation for the observed process and its spectrum.

2. First-order autoregression

Consider a continuous process y that is sampled once every unit of time.[2] We take as given the unconditional mean μ and the unconditional autocovariance function, $\gamma(j)$ for $j = 0, 1, 2, \ldots$, where $\gamma(0)$ is the unconditional variance. The autocorrelation function is given by $\rho(j) = \gamma(j)/\gamma(0)$. We assume that the discretely-sampled process is a stationary AR(1), in which case, the autocorrelation function is completely determined by $\rho(1)$: $\rho(j) = \rho(1)^j$ for $j = 2, 3, 4, \ldots$. We can write the dynamics of the observed time series as

$$y(t+h) = \mu + \rho(h)\,(y(t) - \mu) + \nu(t,h), \tag{2.1}$$

for $h = 1$. The "error term" $\nu(t,1)$ is normally distributed with mean zero and constant variance

$$\sigma_\nu^2(h) = (1 - \rho(h)^2)\,\gamma(0), \tag{2.2}$$

for $h = 1$. In addition, $\nu(t,1)$ is serially uncorrelated and uncorrelated with $y(t)$. If the autocorrelation coefficient is positive, $\rho(1) > 0$, then we can extend (2.1) and (2.2) to all $h > 0$ by letting $\rho(h) = e^{-\kappa h}$, where $\kappa > 0$. The expectation of $y(t+h)$ conditional on $y(t)$ is

$$E[y(t+h) \mid y(t)] = \mu + \rho(h)\,\big(y(t) - \mu\big). \tag{2.3}$$

The extension is a parsimonious way to provide forecasts for non-integer horizons that are consistent with those for integral horizons.

Let $p(x,h,z,t)$ be the probability density of $y(t+h) = x$ given $y(t) = z$. The conditional density implied by (2.1)–(2.3) is

$$p(x,h,z,t) = \mathcal{N}\Big(x,\ \mu + \rho(h)\,\big(z - \mu\big),\ (1 - \rho(h)^2)\,\gamma(0)\Big), \tag{2.4}$$

where

$$\mathcal{N}(x,m,v) = (2\,\pi\,v)^{-1/2}\,e^{-\frac{(x-m)^2}{2\,v}}$$

is the normal density with mean m and variance v. The transition density for a Markovian variable satisfies the Chapman–Kolmogorov equation:

$$p(x, h_1 + h_2, y, t) = \int_{-\infty}^{\infty} p(x, h_2, z, t + h_1)\, p(z, h_1, y, t)\, dz, \quad \text{for all } h_1, h_2 > 0. \tag{2.5}$$

The probability density given by (2.4) satisfies (2.5) for $\rho(h) = e^{-\kappa h}$. Therefore, y follows and AR(1) for any $h > 0$. In particular, let y be an Ito process with dynamics given by

$$dy(t) = \kappa\,(\mu - y(t))\,dt + \sqrt{2\,\gamma(0)\,\kappa}\,dW(t), \tag{2.6}$$

[2] Assume y is a "stock" variable. There is some discussion of "flow" variables in Appendix C.

where W is a Wiener process. This process satisfies the conditional probability density above.[3]

In order to accommodate negative first-order autocorrelation, define

$$q(h) = e^{-\kappa\, h} \left(\cos(\pi\, h) + \frac{\psi\, \sin(\pi\, h)}{\pi} \right), \tag{2.7}$$

and where ψ is a free parameter.[4] Now suppose (2.1)–(2.3) hold, where the autocorrelation function is given by $\rho(h) = q(h)$. See Figures 1 and 2 for a plots of $q(h)$ and $1 - q(h)^2$ for $\psi = -\kappa,\ 0,\ \kappa$. Note the following three properties of $q(h)$. First, $q(1) = -e^{-\kappa} < 0$. Second, $q(h) = 0$ for $h = \delta, 1 + \delta, 2 + \delta, \ldots$, where

$$\delta = \frac{1}{\pi} \arccos \left(\frac{-\psi}{\sqrt{\pi^2 + \psi^2}} \right). \tag{2.8}$$

For example, for $\psi = 0$, $\delta = 1/2$. Third, $dq(h)/dh|_{h=0} = \psi - \kappa$.

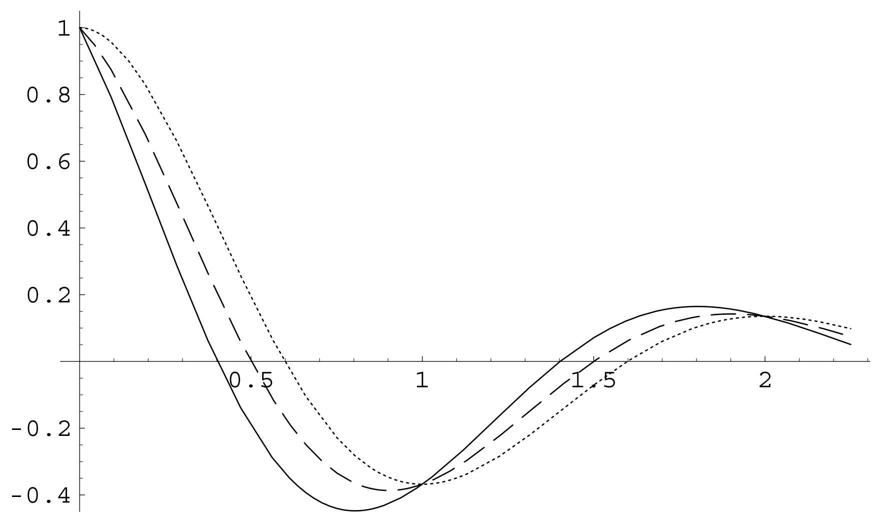

FIGURE 1. Autocorrelation function $q(h)$. The solid curve shows $\psi = -\kappa$, the dashed curve shows $\psi = 0$, and the dotted curve shows $\psi = \kappa$.

The autocorrelation function $q(h)$ satisfies all of the conditions for an AR(1) for $h = 1$. However, (2.5) is not satisfied except for integral h_j, which implies $\nu(t, h)$ is serially correlated for non-integral h. Consequently, y is not an AR(1) at any other sampling frequency. Moreover, the failure to satisfy (2.5) demonstrates that this conditional probability cannot be generated by a single Markovian state variable.

By allowing the continuous-time dynamics for y to depend on an unobserved state variable, a model that generates this conditional probability can be found. In

[3]See, for example, Harvey (1989, pp. 480–482).

[4]As we will see below, ψ is restricted to the interval $[-\kappa, \kappa]$.

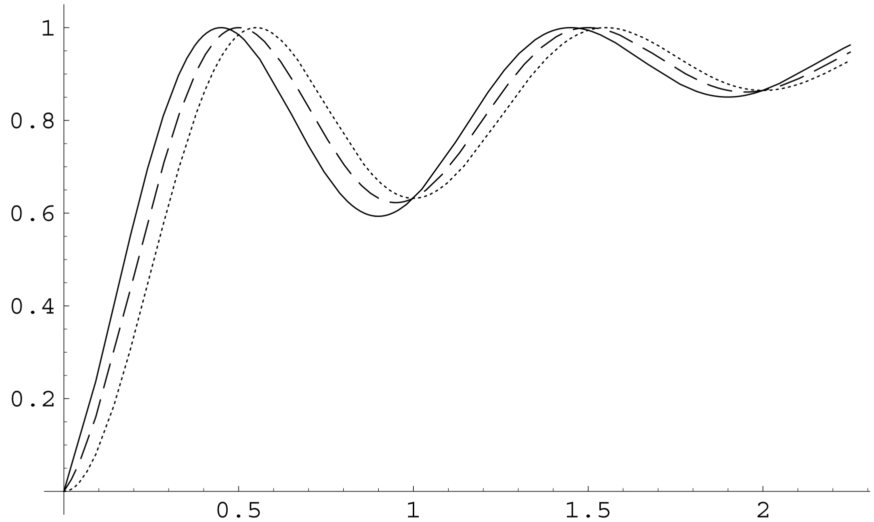

FIGURE 2. Variance of $\nu(t,h)$ is proportional to $1 - q(h)^2$; see (2.2). The solid curve shows $\psi = -\kappa$, the dashed curve shows $\psi = 0$, and the dotted curve shows $\psi = \kappa$.

particular, let z be the unobserved variable and let the joint dynamics of y and z be given by

$$dy(t) = \big\{(\kappa - \psi)\,(\mu - y(t)) + \xi\, z(t)\big\}\, dt + \sqrt{2\,\gamma(0)\,(\kappa - \psi)}\, dW_1(t) \tag{2.9a}$$

$$dz(t) = \left\{ \left(\frac{\pi^2 + \psi^2}{\xi}\right) (\mu - y(t)) - (\kappa + \psi)\, z(t) \right\} dt + \sqrt{\frac{2\,\gamma(0)\,(\kappa + \psi)\,(\pi^2 + \psi^2)}{\xi^2}}\, dW_2(t), \tag{2.9b}$$

where W_1 and W_2 are independent Wiener processes and $\xi \neq 0$ is a nuisance parameter. As we will see, ξ has no observable implications for y (or the coherence between y and z). In the next section I show that (2.9) produces the probability density (2.4) where $\rho(h) = q(h)$.

The only free parameter with observable implications is ψ. We briefly consider three cases. First, for $\psi = \kappa$, the diffusion term for y vanishes, in which case y has finite variation and is (equivalently) mean-square differentiable. This is the case that fits into Bergstrom's framework (with $\xi = 1$). Second, the greatest symmetry between the dynamics of y and z is produced with $\psi = 0$ (and $\xi = \pi$). This is the specification Harvey (1989) chooses to model the cyclic component. Third, for $\psi = -\kappa$, the diffusion term for z vanishes, in which case z has finite variation. As I show in Section 4, the coherence between y and z is identical for $\psi = \pm\kappa$.

3. The continuous-time Markovian structure

In this section, we build solution (2.9) from scratch. Consider a system of two Ito processes that is linear and Gaussian. In particular, let

$$dX(t) = \big(a + b\,X(t)\big)\,dt + \Sigma\,dW(t), \tag{3.1}$$

where

$$X(t) = \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} \qquad \text{and} \qquad W(t) = \begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix},$$

and where

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \qquad \text{and} \qquad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}.$$

Assume b is invertible and let $m = -b^{-1}a$. Given this setup, the distribution of $X(t+h)$ conditional on $X(t)$ is Gaussian, and as such it is completely determined by its conditional mean and conditional variance.

In discrete time, this model produces a vector autoregression process for any time step h:[5]

$$X(t+h) = \alpha(h) + \beta(h)\,X(t) + \varepsilon(t,h), \tag{3.2}$$

where

$$\alpha(h) = \big(I - \beta(h)\big)\,m \qquad \text{and} \qquad \beta(h) = e^{b\,h},$$

and the error term $\varepsilon(t,h)$ is normally distributed with mean zero and variance-covariance matrix

$$V(h) = \int_{v=0}^{h} \beta(h-v)\,\Sigma\,\Sigma^\top \beta(h-v)^\top\,dv.$$

The error term is serially uncorrelated.[6] The stationarity of X implies $\lim_{h\to\infty}\ \beta(h) = 0$, $\lim_{h\to\infty}\ \alpha(h) = m$, and $V(\infty) = \lim_{h\to\infty}\ V(h)$. Note that $V(\infty)$ is the unconditional variance-covariance matrix of X, and therefore one condition that must be satisfied is $V_{11}(\infty) = \gamma(0)$.

By choosing $a_j = -\mu\,b_{1j}$ we force $m_1 = \mu$ and $m_2 = 0$, which allows us to write the first line of (3.2) as

$$y(t+1) = \mu + \beta_{11}(h)\,(y(t) - \mu) + \nu(t,h), \tag{3.3}$$

where

$$\nu(t,h) = \varepsilon_1(t,h) + \beta_{12}(h)\,z(t). \tag{3.4}$$

By construction, $\nu(t,h)$ has mean zero. If in addition $z(t)$ is unconditionally uncorrelated with $y(t)$ (*i.e.*, if $V_{12}(\infty) = 0$), then[7]

$$E[y(t+h) \mid y(t)] = \mu + \beta_{11}(h)\,(y(t) - \mu). \tag{3.5}$$

Nevertheless, the presence of $z(t)$ makes $\nu(t,h)$ serially correlated. If $\beta_{12}(1) = 0$ and $\beta_{11}(1) < 0$, then y will follow a univariate AR(1) with negative first-order autocorrelation for $h = 1$.

[5]This solution is presented in Harvey (1989) and is developed step-by-step in Appendix A.

[6]See Appendix A.

[7]This assumption has no observable implications. See Appendix B.

In order to satisfy the requirements we have laid out in the previous paragraph, we need a stable, oscillatory system. To this end, let us reparameterize the coefficients in b as follows:

$$b_{11} = \psi - \kappa, \quad b_{12} = \xi, \quad b_{21} = -\left(\frac{\pi^2+\psi^2}{\xi}\right), \quad \text{and} \quad b_{22} = -(\kappa+\psi), \tag{3.6}$$

where $\kappa > 0$ and $\xi \neq 0$. With this parameterization, the eigenvalues of b are $-\kappa \pm \pi\, i$, where $i = \sqrt{-1}$, and

$$\beta(h) = e^{-\kappa h}\left(\cos(\pi h)\, I + \frac{\sin(\pi h)}{\pi}\begin{pmatrix} \psi & \xi \\ -\left(\frac{\pi^2+\psi^2}{\xi}\right) & -\psi \end{pmatrix}\right), \tag{3.7}$$

where I is the 2×2 identity matrix. Note that $\beta(1) = -e^{-\kappa}\, I$, so that $\beta_{11}(1) < 0$ and $\beta_{12}(1) = 0$ as desired.[8] In particular, $\beta_{11}(h) = q(h)$ as given in (2.7). The conditions $V_{11}(\infty) = \gamma(0)$ and $V_{12}(\infty) = 0$ can be reduced to $\sigma_{12} = \sigma_{21} = 0$,

$$\sigma_{11} = \sqrt{2\,\gamma(0)\,(\kappa-\psi)} \qquad \text{and} \qquad \sigma_{22} = \sqrt{\frac{2\,\gamma(0)\,(\kappa+\psi)\,(\pi^2+\psi^2)}{\xi^2}}.$$

The requirement that σ_{11} and σ_{22} be real restricts the range of ψ to $-\kappa \leq \psi \leq \kappa$. With this parameterization we have

$$V_{11}(h) = \gamma(0)\left(1 - \frac{\pi^2 + \psi^2\,(1-\cos(2\,h\,\pi)) + \pi\,\psi\,\sin(2\,h\,\pi)}{e^{2\,h\,\kappa}\,\pi^2}\right)$$

$$V_{12}(h) = \frac{2\,\gamma(0)\,\psi\,\left(\pi^2+\psi^2\right)\,\sin(h\,\pi)^2}{\xi\,e^{2\,h\,\kappa}\,\pi^2}$$

$$V_{22}(h) = \frac{\pi^2+\psi^2}{\xi^2}\left(V_{11}(h) + \frac{2\,\gamma(0)\,\psi\,\sin(2\,h\,\pi)}{e^{2\,h\,\kappa}\,\pi}\right).$$

Given this parameterization, the variance of $\nu(t,h)$ is

$$V_{11}(h) + \beta_{12}(h)^2\, V_{22}(\infty) = \left(1 - q(h)^2\right)\gamma(0). \tag{3.8}$$

Equation (3.8) confirms that (3.3) has all the properties set out in Section 2. Finally, note that ξ does not appear in either the conditional expectation (3.5) or the conditional variance (3.8).

4. ARMA representation

We have demonstrated that (2.9) produces the desired observable properties for y. In particular, the conditional expectation $E[y(t+h) \mid y(t)]$ is given by (2.3) where $\rho(h) = q(h)$ as given by (2.7). However, the error term $\nu(t,h)$ in (2.1) is serially correlated in general so that lagged values of y may be used to improve forecasts. In this section, I complete the analysis by deriving the discrete-time ARMA process for y. We will see that for small h, the ARMA process for y for $\psi = \kappa$ is quite

[8] More generally one can let $b_{21} = -\left(\lambda^2+\psi^2\right)/\xi$, where $\lambda > 0$, in which case the eigenvalues of b are $-\kappa \pm \lambda\, i$ and $\beta(\lambda/\pi) = -e^{-\kappa\,\lambda/\pi}\, I$. Setting $\psi = 0$ and $\xi = \lambda$, $\lim_{\lambda\to\infty}\ \beta(h) = e^{-\kappa\, h}\, I$, showing the positive AR model is the limit of the negative AR model as the frequency of the cycles goes to infinity. See Harvey (1989, pp. 487–488) on this point as well.

different than for $\psi < \kappa$ and yet the coherence between y and z is identical for $\psi = \pm\kappa$.

We can write (3.2) as

$$\Phi(L_h)\, X(t+h) = \alpha(h) + \varepsilon(t,h), \tag{4.1}$$

where

$$\Phi(L_h) = I - \beta(h)\, L_h \qquad \text{and} \qquad L_h^j\, x(s) = x(s - j\,h).$$

We now derive the univariate representation for y in the time domain. Let $\Phi^\dagger(L_h)$ denote the adjoint matrix of $\Phi(L_h)$ and $|\Phi(L_h)|$ denote the determinant of $\Phi(L_h)$:

$$\Phi^\dagger(L_h) = I - \begin{pmatrix} \beta_{22}(h) & -\beta_{12}(h) \\ -\beta_{21}(h) & \beta_{11}(h) \end{pmatrix} L_h$$

and

$$|\Phi(L_h)| = 1 - \varphi_1(h)\, L_h - \varphi_2(h)\, L_h^2,$$

where $\varphi_1(h) = 2\, e^{-\kappa\, h}\, \cos(\pi\, h)$ and $\varphi_2(h) = -e^{-2\,\kappa\, h}$. See Figure 3.

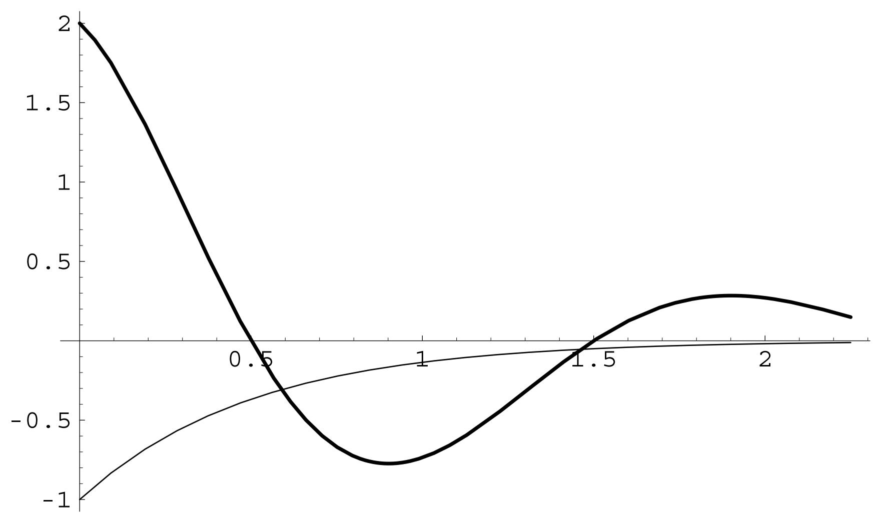

FIGURE 3. Autocorrelation coefficients: $\varphi_1(h)$ is the thick solid line and $\varphi_2(h)$ is the thin solid line.

We can write the discretely-sampled system as

$$|\Phi(L_h)|\, X(t+h) = \hat{\alpha}(h) + \Phi^\dagger(L_h)\, \varepsilon(t,h), \tag{4.2}$$

where $\hat{\alpha}(h) = \Phi^\dagger(1)\, \alpha(h)$. For y we have

$$|\Phi(L_h)|\, y(t+h) = \hat{\alpha}_1(h) + (1 - \beta_{22}(h)\, L_h)\, \varepsilon_1(t,h) + \beta_{12}(h)\, L_h\, \varepsilon_2(t,h), \tag{4.3}$$

which is an ARMA(2,1). Note that $|\Phi(L_h)|$ depends only on κ and so is completely determined by $\rho(1)$. The free parameter affects the process solely through the MA component. Regarding the MA component, there are two special cases of note. First, $\beta_{12}(h) = 0$ for integer h, and the resulting MA polynomial has a common factor in the AR polynomial, reducing the dynamics to the AR(1) case.[9] Second, $\beta_{22}(h) = 0$ for $h = \delta, 1+\delta, 2+\delta, \ldots$, where δ is given in (2.8), so that the remaining error is serially uncorrelated, producing a pure AR(2).

We can compute the MA coefficient as follows. Referring to (4.3), define

$$\begin{aligned}\zeta(t,h) &:= |\Phi(L_h)|\, y(t+h) - \hat{\alpha}(h)\\ &= \varepsilon_1(t,h) - \beta_{22}(h)\,\varepsilon_1(t-h,h) + \beta_{12}(h)\,\varepsilon_2(t-h,h).\end{aligned}$$

The variance and first-order covariance of $\zeta(t,h)$ are

$$\begin{aligned}\gamma_\zeta^h(0) &= \left(1+\beta_{22}(h)^2\right) V_{11}(h) + \beta_{12}(h)^2\, V_{22}(h) - 2\,\beta_{12}(h)\,\beta_{22}(h)\, V_{12}(h)\\ &= \gamma(0)\left(1 - e^{-4\,h\,\kappa} - \frac{2\,\psi\, e^{-2\,h\,\kappa}\,\sin(2\,h\,\pi)}{\pi}\right)\\ \gamma_\zeta^h(1) &= -\beta_{22}(h)\,V_{11}(h) + \beta_{12}(h)\,V_{12}(h)\\ &= \frac{2\,\gamma(0)\,e^{-2\,h\,\kappa}\left(\psi\,\cosh(h\,\kappa)\,\sin(h\,\pi) - \pi\,\cos(h\,\pi)\,\sinh(h\,\kappa)\right)}{\pi}.\end{aligned}$$

Figures 4 and 5 plot $\gamma_\zeta^h(0)$ and $\gamma_\zeta^h(1)$ versus h for the three values of ψ. For small h, the behavior of both $\gamma_\zeta^h(0)$ and $\gamma_\zeta^h(1)$ for $\psi = \kappa$ is quite different than for $\psi < \kappa$:

$$\gamma_\zeta^h(0) = \begin{cases} 4\,(\kappa-\psi)\,\gamma(0)\,h + \mathcal{O}(h^2) & \psi < \kappa\\ \frac{8}{3}\,\kappa\,(\pi^2+\kappa^2)\,\gamma(0)\,h^3 + \mathcal{O}(h^4) & \psi = \kappa\end{cases}$$

and

$$\gamma_\zeta^h(1) = \begin{cases} -2\,(\kappa-\psi)\,\gamma(0)\,h + \mathcal{O}(h^2) & \psi < \kappa\\ \frac{2}{3}\,\kappa\,(\pi^2+\kappa^2)\,\gamma(0)\,h^3 + \mathcal{O}(h^4) & \psi = \kappa.\end{cases}$$

We can write $\zeta(t,h)$ in MA form:

$$\zeta(t,h) = (1+\theta(h)\,L_h)\;\eta(t,h),$$

where $\eta(t,h)$ is normally distributed with mean zero and variance $\sigma_\eta^2(h)$. Out goal is to find expressions for $\theta(h)$ and $\sigma_\eta^2(h)$. We can solve

$$\gamma_\zeta^h(0) = \left(1+\theta(h)^2\right)\sigma_\eta^2(h) \qquad \text{and} \qquad \gamma_\zeta^h(1) = \theta(h)\,\sigma_\eta^2(h)$$

for $\theta(h)$ and $\sigma_\eta^2(h)$. In Figures 6 and 7, $\theta(h)$ and $\sigma_\eta^2(h)$ are plotted versus h for $\psi = -\kappa, 0, \kappa$. For small h, the behavior of $\theta(h)$ and $\sigma_\eta^2(h)$ for $\psi = \kappa$ is quite different than for $\psi < \kappa$ as well:

$$\theta(h) = \begin{cases} -1 + \mathcal{O}(h^{1/2}) & \psi < \kappa\\ 2 - \sqrt{3} + \mathcal{O}(h) & \psi = \kappa\end{cases}$$

and

$$\sigma_\eta^2(h) = \begin{cases} 2\,(\kappa-\psi)\,h + \mathcal{O}(h^{3/2}) & \psi < \kappa\\ \frac{2}{3}\,(2+\sqrt{3})\,\kappa\,(\pi^2+\kappa^2)\,h^3 + \mathcal{O}(h^4) & \psi = \kappa.\end{cases}$$

[9] See (3.7) for the parameterization of $\beta(h)$.

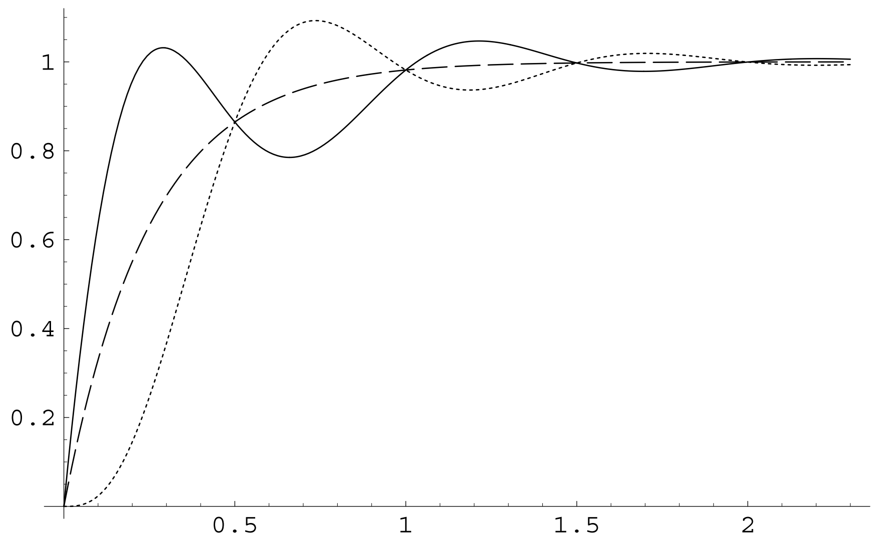

FIGURE 4. $\gamma_\zeta^h(0)$ is plotted for three values of ψ (for $\kappa = \gamma(0) = 1$). The solid curve shows $\psi = -\kappa$, the dashed curve shows $\psi = 0$, and the dotted curve shows $\psi = \kappa$.

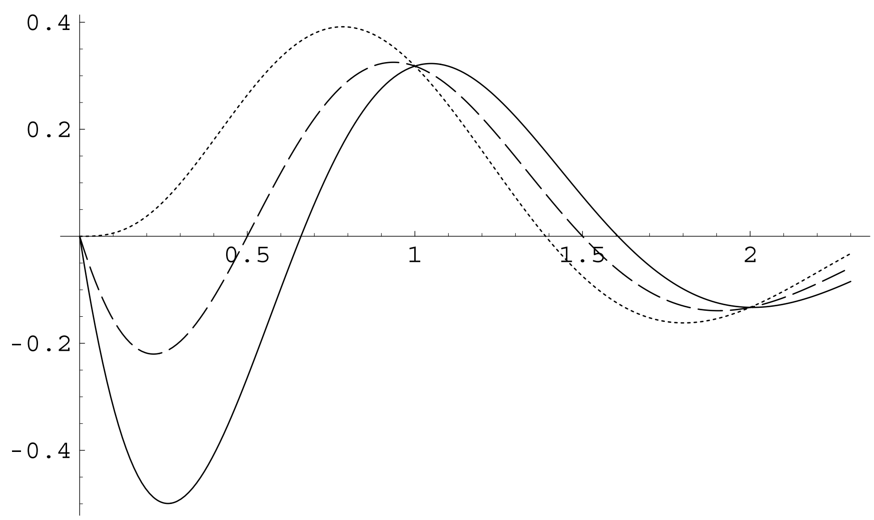

FIGURE 5. $\gamma_\zeta^h(1)$ is plotted for three values of ψ (for $\kappa = \gamma(0) = 1$). The solid curve shows $\psi = -\kappa$, the dashed curve shows $\psi = 0$, and the dotted curve shows $\psi = \kappa$.

The spectrum. The spectrum provides a concise way to summarize the time-series properties of y and its relation to the omitted variable z for various values of the

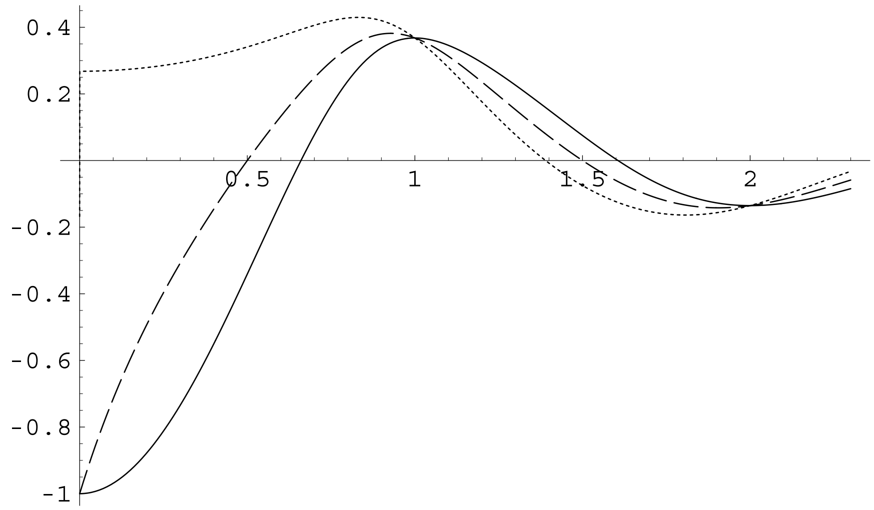

FIGURE 6. $\theta(h)$ is plotted for three values of ψ (for $\kappa = \gamma(0) = 1$). The solid curve shows $\psi = -\kappa$, the dashed curve shows $\psi = 0$, and the dotted curve shows $\psi = \kappa$.

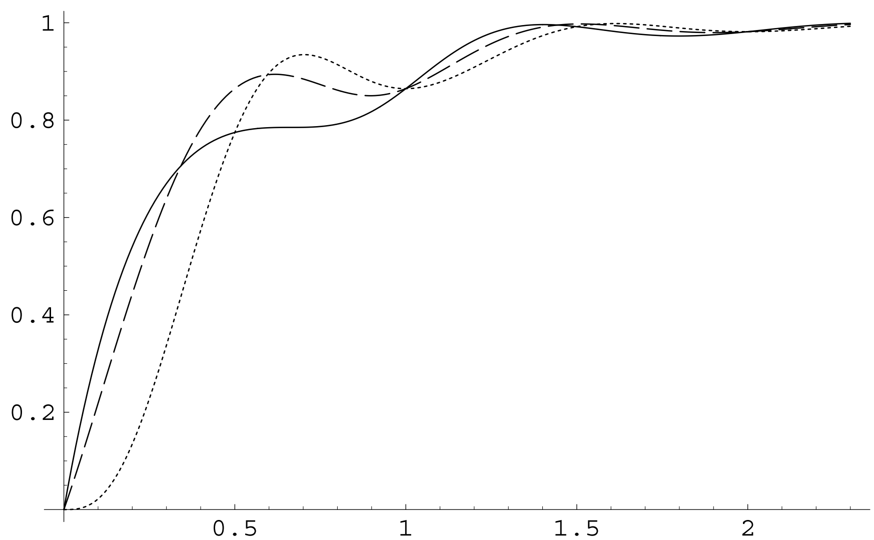

FIGURE 7. $\sigma_\eta^2(h)$ is plotted for three values of ψ (for $\kappa = \gamma(0) = 1$). The solid curve shows $\psi = -\kappa$, the dashed curve shows $\psi = 0$, and the dotted curve shows $\psi = \kappa$.

sampling frequency h.[10] The spectrum of finitely-sampled X is given by[11]

$$f_X(\omega)(h) = \Phi\left(e^{-i\,\omega}\right)^{-1} V(h) \left(\Phi\left(e^{i\,\omega}\right)^{-1}\right)^{\top} = \begin{pmatrix} f_y(\omega)(h) & f_{yz}(\omega)(h) \\ f_{zy}(\omega)(h) & f_z(\omega)(h) \end{pmatrix}. \tag{4.4}$$

[10]The spectrum of X itself, which does not depend on the sampling horizon, is less informative.

[11]See Hamilton (1994).

Here is $2\, f_y(\omega)(h)/\gamma(0)$:[12]

$$\frac{2\, \cos(\omega)\big(\psi\, \cosh(h\,\kappa)\, \sin(h\,\pi) - \pi\, \cos(h\,\pi)\, \sinh(h\,\kappa)\big) - \psi\, \sin(2\,h\,\pi) + \pi\, \sinh(2\,h\,\kappa)}{2\,\pi^2\, \left((\cos(h\,\pi) - \cos(\omega)\, \cosh(h\,\kappa))^2 + \sin(\omega)^2 \sinh(h\,\kappa)^2\right)}.$$

For $h = 1$,[13]

$$\frac{2\, f_y(\omega)(1)}{\gamma(0)} = \frac{\sinh(\kappa)}{\pi\,(\cosh(\kappa) + \cos(\omega))}.$$

Figure 8 shows the normalized spectrum of y for $\psi = 0$. For comparison, the normalized spectrum for the three values of ψ is plotted in Figure 9 for $h = 1/4$. For all appropriate values of the parameters, $\lim_{\kappa\to\infty}\, 2\,\pi\, f_y(\omega)(h)/\gamma(0) = 1$, which shows that y converges to white noise as κ increases without bound.

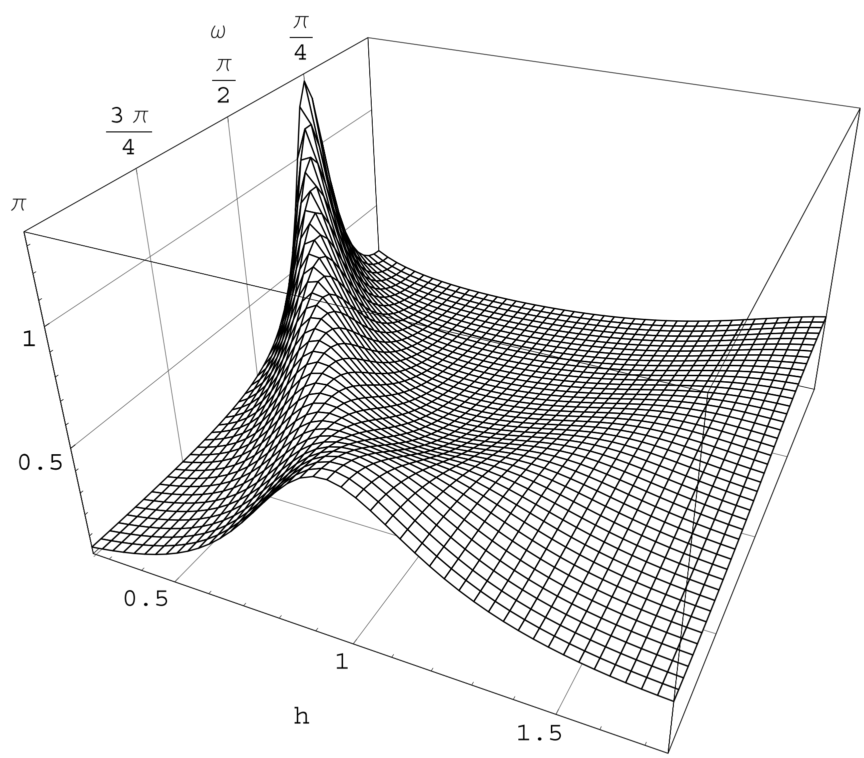

FIGURE 8. The normalized spectrum of y, $2\, f_y(\omega)(h)/\gamma(0)$, for $\psi = 0$.

Here is the cross-spectrum $f_{yz}(\omega)(h)/\gamma(0)$:

$$\frac{-i\, \left(\pi^2 + \psi^2\right)\, \sin(h\,\pi)\, \sin(\omega)\, \sinh(h\,\kappa)}{\xi\,\pi^2\, (1 + \cos(2\,h\,\pi) + \cos(2\,\omega) - 4\, \cos(h\,\pi)\, \cos(\omega)\, \cosh(h\,\kappa) + \cosh(2\,h\,\kappa))}.$$

[12]Note that $\gamma(0) = 2\int_0^\pi f_y(\omega)(h)\, d\omega$ for $h > 0$.

[13]For comparison, the normalized spectrum when y is positively autocorrelated is

$$\frac{2\, f_y(\omega)(h)}{\gamma(0)} = \frac{\sinh(\kappa\, h)}{\pi\,(\cosh(\kappa\, h) - \cos(\omega))}.$$

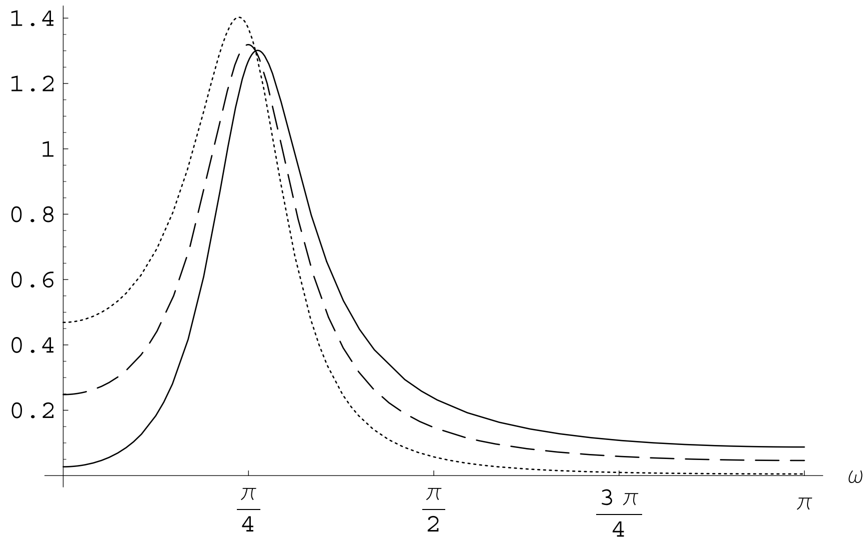

FIGURE 9. The normalized spectrum of y, $2\, f_y(\omega)(1/4)/\gamma(0)$. The solid curve shows $\psi = -\kappa$, the dashed curve shows $\psi = 0$, and the dotted curve shows $\psi = \kappa$.

Note that $2 f_{yz}(\omega)(1) = 0$. The coherence is given by

$$c_{yz}(\omega)(h) = \frac{|f_{yz}(\omega)(h)|^2}{f_y(\omega)(h)\, f_z(\omega)(h)}.$$

The general expression for the coherence is somewhat complicated. Note the following properties of the coherence. It is independent of ξ, and it is zero for integer h. As a function of ψ, it is symmetric around $\psi = 0$ where it reaches its minimum. For $\psi = 0$,

$$c_{yz}(\omega)(h) = \left(\frac{\sin(h\,\pi)\sin(\omega)}{\cosh(h\,\kappa) - \cos(h\,\pi)\,\cos(\omega)} \right)^2.$$

Figure 10 shows the coherence for $\psi = 0$. For comparison, the coherence for the three values of ψ is plotted in Figure 11 for $h = 1/4$.

APPENDIX A. DERIVATION OF DISCRETE-TIME VECTOR AR PROCESS

By construction,

$$X(t+h) = X(t) + \int_{s=0}^{h} dX(t+s). \tag{A.1}$$

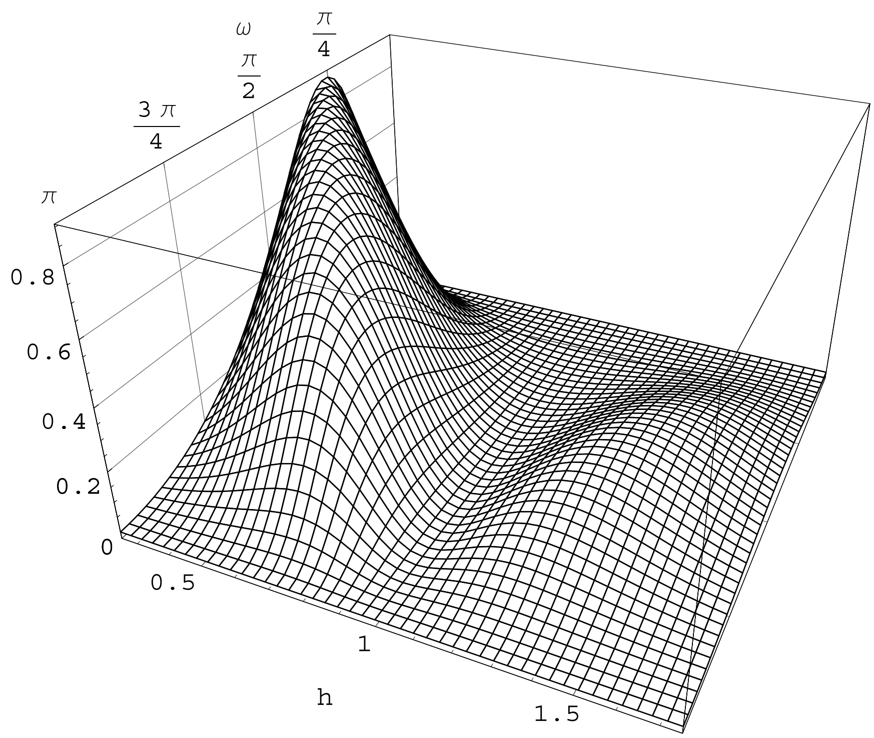

FIGURE 10. The coherence of y and z, $c_{yz}(\omega)(h)$, for $\psi = 0$.

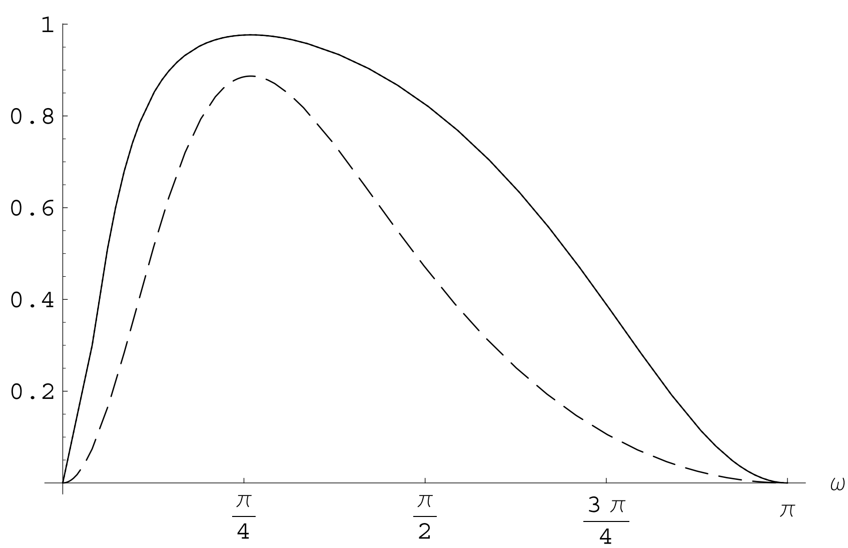

FIGURE 11. The coherence of y and z, $c_{yz}(\omega)(1/4)$. The solid curve shows $\psi = \pm\kappa$ and the dashed curve shows $\psi = 0$.

Let $X(t,h) = E[X(t+h) \mid X(t)]$. Taking conditional expectations of both sides of (A.1) produces

$$\begin{aligned} X(t,h) &= X(t) + E\left[\int_{s=0}^{h} dX(t+s) \mid X(t)\right] \\ &= X(t) + \int_{s=0}^{h} E\left[dX(t+s) \mid X(t)\right] \\ &= X(t) + \int_{s=0}^{h} E\left[\big(a + b\,X(s)\big)\,ds \mid X(t)\right] \\ &= X(t) + \int_{s=0}^{h} \big(a + b\,X(t,s)\big)\,ds. \end{aligned}$$

Thus $X(t,h)$ satisfies a system of first-order ODEs with constant coefficients:

$$\frac{d}{dh}X(t,h) = a + b\,X(t,h) \qquad \text{subject to } X(t,0) = X(t).$$

The solution is

$$X(t,h) = \alpha(h) + \beta(h)\,X(t),$$

where

$$\alpha(h) = \big(I - \beta(h)\big)\,m \qquad \text{and} \qquad \beta(h) = e^{b\,h}.$$

Now consider the process for the conditional expectation of $X(T)$ for some fixed $T \geq t$. In other words, apply Ito's lemma to $X(t,T-t) = \alpha(T-t) + \beta(T-t)\,X(t)$:

$$dX(t,T-t) = \mu(t,T-t)\,dt + \Sigma(t,T-t)\,dW(t),$$

where $\Sigma(t,h) = \beta(h)\,\Sigma$ and

$$\begin{aligned} \mu(t,h) &= \beta(h)\big(a + b\,X(t)\big) - \big(\alpha'(h) + \beta'(h)\,X(t)\big) \\ &= \big\{\beta(h)\,a - \alpha'(h)\big\} + \big\{\beta(h)\,b - \beta'(h)\big\}\,X(t) \\ &= 0. \end{aligned}$$

Let $T = t + h$. Then we can define the "error" term:

$$\begin{aligned} \varepsilon(t,h) &= X(t+h) - X(t,h) \\ &= \int_{v=0}^{h} dX(t+v, h-v) \\ &= \int_{v=0}^{h} \big(\beta(h-v)\,\Sigma\big)\,dW(t+v). \end{aligned} \tag{A.2}$$

The error term is normally distributed with mean zero and variance-covariance matrix

$$V(h) = \int_{v=0}^{h} \beta(h-v)\,\Sigma\,\Sigma^\top \beta(h-v)^\top\,dv.$$

Note that since there is no overlap,

$$\mathrm{Cov}[\varepsilon(t,h),\ \varepsilon(t+j\,h,h)] = 0 \qquad \text{for } j = 1,2,3,\ldots.$$

In other words, $\varepsilon(t,h)$ is serially uncorrelated.

Appendix B. Unconditional correlation

In this section I show that the assumption that y and z are unconditionally uncorrelated is not restrictive in the sense that (2.3) holds where $\rho(h)$ is given by (2.7) even when y and z are unconditionally correlated. To allow for a more general correlation structure, let

$$\Sigma = \begin{pmatrix} \sigma_{11} & 0 \\ \sigma_{21} & \sigma_{22} \end{pmatrix}.$$

With $\beta(h)$ given by (3.7), we have

$$V_{12}(\infty) = \frac{2\,(\kappa - \psi)\, V_{11}(\infty) + \sigma_{11}^2}{2\,\xi}.$$

We can decompose z into two orthogonal parts as follows. Let $z(t) = c_0 + c_1\, y(t) + u(t)$, where $c_0 = m_2 - c_1\, m_1$, $c_1 = V_{12}(\infty)/V_{11}(\infty)$ and $u(t)$ and $y(t)$ are unconditionally uncorrelated. The variance of $u(t)$ is

$$V_{22}(\infty) - c_1^2\, V_{11}(\infty) = \frac{\det(V(\infty))}{V_{11}(\infty)}.$$

We can write

$$\begin{aligned} Y(t+h) &= \big(\alpha_1(h) + \beta_{11}(h)\, c_0\big) + \big(\beta_{11}(h) + c_1\, \beta_{12}(h)\big)\, Y(t) + \nu(t,h) \\ &= \tilde{\alpha}_{11}(h) + \tilde{\beta}_{11}(h)\, y(t) + \nu(t,h), \end{aligned} \tag{B.1}$$

where $\nu(t,h) = \varepsilon_1(t,h) + \beta_{12}(h)\, u(t)$ is mean zero with variance

$$V_{11}(h) + \beta_{12}(h)^2\, \frac{\det(V(\infty))}{V_{11}(\infty)}.$$

The autocorrelation function is

$$\tilde{\beta}_{11}(h) = e^{-\kappa\, h} \left(\cos(\pi\, h) + \frac{\tilde{\psi}\, \sin(\pi\, h)}{\pi} \right),$$

where

$$\tilde{\psi} = \kappa - \frac{\sigma_{11}^2}{2\,\gamma(0)}.$$

Clearly, $\tilde{\psi} \leq \kappa$. One can show that $\sigma_{22}^2 \geq 0$ implies $\tilde{\psi} \geq -\kappa$.

Appendix C. Flow variables

We now consider the discrete-time process for “flow” variables, where

$$S_h(t) = \int_{s=t-h}^{t} X(s)\, ds.$$

We can write

$$\begin{aligned} S_h(t+h) &= \int_{s=t}^{t+h} X(s)\,ds \\ &= \int_{s=t}^{t+h} \Big(\alpha(h) + \beta(h)\,X(s-h) + \varepsilon(s-h,h)\Big)\,ds \\ &= \alpha(h)\,h + \beta(h)\,S_h(t) + u(t-h,t+h), \end{aligned}$$

where (see Appendix A)

$$u(t-h,t+h) = \int_{s=t}^{t+h} \varepsilon(s-h,h)\,ds = \int_{s=t}^{t+h}\int_{v=0}^{h} \big(\beta(h-v)\,\Sigma\big)\,dW(s-h+v). \quad \text{(C.1)}$$

The index on the Brownian in (C.1) runs from $t-h$ to $t+h$, so that $u(t-h,t+h)$ is correlated with $u(t,t+2\,h)$. Thus S_h follows an ARMA(1,1).[14] Nevertheless, with the specialized parameterization, the forecast of $S_{1h}(t+h)$ conditional on $S_{1h}(t)$ is

$$E[S_{1h}(t+h) \mid S_{1h}(t)] = \mu\,h + q(h)\,(S_{1h}(t) - \mu\,h)\,,$$

where $q(h)$ is given by (2.7).

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[14] Reversing the order of integration,

$$\begin{aligned} u(t-h,t+h) &= \int_{v=0}^{h} \big(\beta(h-v)\,\Sigma\big)\left(\int_{s=t}^{t+h} dW(s-h+v)\right)dv \\ &= \int_{v=0}^{h} \big(\beta(h-v)\,\Sigma\big)\,w(v,h)\,dv, \end{aligned}$$

where $w(v,h) = W(t+v) - W(t+v-h)$ is normally distributed with mean zero and variance–covariance matrix

$$E\left[w(v,h)\,w(v+u,h)^\top\right] = (h-u)\,I \qquad \text{for } 0 \le u \le h-v.$$