APPARENT ARBITRAGE

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Still a bit incomplete.

Abstract. The doubling strategy does not produce an arbitrage in the space of signed measures equipped with the weak* topology. (A signed measure represents a payout.) The absence of arbitrage opportunities is guaranteed by the existence of a valuation operator (a strictly positive continuous linear functional). Nevertheless, the sequence of signed measures generated by the doubling strategy produces what appears to be an arbitrage from the perspective of convergence in measure of the corresponding sequence of Radon–Nikodym derivatives taken with respect to a fixed positive numeraire measure.

An apparent arbitrage is not what it seems: It appears to be an arbitrage from the perspective of convergence in measure, but it is not arbitrage from the perspective of weak-* convergence. (By contrast, an approximate arbitrage converges in the norm topology.)

Introduction [incomplete]

In their seminal paper, Harrison and Kreps (1979, p. 400) refer to

> the well known doubling strategy by which one is sure to win at roulette: Bet $1 on red, and keep doubling your bets until red comes out. To effect this strategy, you must be able to bet a countable number of times, although you will only bet a finite number of times in any particular state.

The doubling strategy is an arbitrage because each bet is costless, while the gain after n bets converges to $1 with probability one as n goes to infinity. Side conditions typically solve this pathology by removing the doubling strategy from the choice set. Note, however, that in describing the situation as a pathology we have adopted the topology of convergence in probability (the topology of almost sure convergence has the same effect). With other topologies the doubling strategy does not necessarily converge to $1, and restricting the strategy choice becomes unnecessary. There is no compelling reason to adopt the topology of convergence in probability; often, in fact, there are compelling reasons not to.

Date: February 25, 2010.

We thank Steve LeRoy, Dan Waggoner, and Freddy Delbaen for useful discussions. The views expressed herein are the authors' and do not necessarily reflect those of the Federal Reserve Bank of Atlanta, the Federal Reserve System, or Bear, Stearns & Co. This paper represents a substantial revision of (and supercedes) "An analysis of the doubling strategy: The countable case."

In this paper we present a framework in which the doubling strategy converges but not to an arbitrage. Although the framework is fairly general, we have not yet extended our results to continuous-time stochastic processes.

Our main point is that the choice of a topology dictates which sequences of payouts are arbitrages, a point made by Kreps (1981), who apparently did not apply it to the case of dynamic security market models.[1] We illustrate it with the traditional example of the doubling strategy cast in the simplest possible setting: A countable state space where the state of the world is simply the number of spins that it takes for the first RED to occur.

Our treatment of the doubling strategy is inspired by the suggestion in Gilles and LeRoy (1997, Section 6.3) that the doubling strategy could be modeled as a payout bubble.[2] Their paper in turn is an outgrowth of Bewley (1972). Our setup also encompasses the example in Back and Pliska (1991).[3] In passing, we mention the approach taken by Delbaen and Schachermayer (1994).

The framework for our approach is that of dual pairs. Aliprantis and Border (1999, p. 163) write[4]

> [W]e are led to the study of *dual pairs* $\langle X, X^* \rangle$ of spaces and their associated *weak topologies*. ... The weak topology on X^* induced by X is called the weak* topology. The most familiar example of a dual pair is probably the pairing of functions and measures—each defines a linear functional via the integral $\int_X f\,d\mu$, which is linear in f for fixed μ, and linear in μ for fixed f. (The weak topology induced on probability measures by this duality with continuous functions is the topology of convergence in distribution that is used in the Central Limit Theorems.) ...
>
> Debreu (1954) introduced dual pairs in economics in order to describe the duality between commodities and prices. According to this interpretation, a dual pair $\langle X, X^* \rangle$ represents the commodity-price duality, where X is the commodity space, X^* is the price space, and $\langle x, x^* \rangle$ is the value of the bundle x at prices x^*. ...

In this paper, we adopt the duality of functions and measures; however we find it convenient to let the space of measures X^* represent the commodity space (i.e.,

[1] In fact, Kreps uses as an example the model we adopt here: the space of signed measures equipped with the weak-* topology. However, the subject matter content (differentiated products markets) of the paper he cites [Mas-Colell (1975)] that use the model is somewhat different from what we propose. See also Jones (1984).

[2] Our analysis of the examples taken from Gilles and LeRoy (1997) is much simpler than what is given there because we have reduced the set of linear functionals relative to those treated in the model in that paper by assuming compactness of X. See below.

[3] Werner (1997) provides extensive treatment of the same static version of their example that we present here. In a related static model Gilles and LeRoy (1998) discuss many of the issues raised by Back and Pliska. Pliska (1997, Chapter 7) provides an introduction to these issues.

[4] We have made some minor adjustments to their notation. In particular, they use X' to denote the topological dual and X^* to denote the algebraic dual, whereas we adopt the more usual notation where X^* denotes the topological dual.

the space of payouts). We equip X^* with the weak* topology. This has the effect (among others) of making X the topological dual of X^*.

Outline of the paper. In Section 1 we present the results from measure theory and functional analysis that we require. To one extend or another, the results are standard. They are presented in a somewhat abstract fashion (i.e., without reference to the content regarding economics and finance). The reader already familiar with such things can refer to this section for notation. Others may be forced to slog through it, wondering how the machinery will be used.

In Section 2 we summarize the Krepsian approach to arbitrage as it bears on our analysis, and we present the skeleton structure of an apparent arbitrage, which we flesh out in the examples in the subsequent sections. Section 3 contains the first set of examples. There is no (explicit) uncertainty here; the examples are intended to allow the reader to dive into weak* convergence with as little overhead as possible.

Section 4 is the heart of the paper. In this section we provide examples in a countable space. The doubling strategy is presented in Example 4.2. In Section 5 we embed the setting of Section 4 in a dynamic securities market, complete with filtration, self-financing trading strategies, and an equivalent martingale measure. In an important sense, there is nothing new in this section: We show that the analysis of arbitrage opportunities in the static model in Section 4 carries over (with suitable changes in terminology) to the standard analysis of arbitrage opportunities in the dynamic setting (up to but not including doubling strategies, of course).

In Section 6 we reverse the roles of our dual pair of spaces in order to make contact with the Back and Pliska (1991) example. In Section **??** we show how our approach can be applied in the Black–Scholes setting. Although our approach here is suggestive, it is not entirely satisfactory owing to the fact that it does not take the underlying stochastic machinery seriously. Consequently, this section provides a jumping-off point for further research into applying the ideas we present here to continuous-time stochastic processes.

1. The duality of functions and measures

Given a compact metric space X, let $C(X) := C(X, \mathbb{R})$ denote the space of continuous functions on X.[5] Equip $C(X)$ with the sup norm topology (denoted $\| \, \|_\infty$), where $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$. $C(X)$ is a Banach lattice. The topological dual space then can be identified with the space of finite signed measures $M(X)$ on the Borel σ-algebra $\mathcal{B}_X := \mathcal{B}(X)$ generated from the Borel sets of X. $M(X)$ is a

[5]Every compact metrizable space can be obtained as a quotient space from the Cantor set. See Appendix A for additional information.

Banach lattice under the total variation norm $\|\mu\| = |\mu|(X) = \mu^+(X) + \mu^-(X)$ for $\mu \in M(X)$.[6,7]

We can equip $M(X)$ with the weak* topology (denoted w^*), in which case its topological dual space is $C(X)$. The pair of spaces $\langle C(X), M(X)\rangle$ form a dual pair with the duality[8]

$$\langle f, \mu\rangle = \int_X f\, d\mu,$$

with $f \in C(X)$ and $\mu \in M(X)$. Given $\mu, \nu \in M(X)$, the separation property of a duality implies

$$\mu = \nu \quad\iff\quad \langle f, \mu\rangle = \langle f, \nu\rangle \qquad \forall\, f \in C(X). \tag{1.1}$$

Consider a sequence $\{\mu_n\}_{n=1}^{\infty} \subset M(X)$ and an element $\mu \in M(X)$. Weak-* convergence is characterized as follows:

$$\mu_n \xrightarrow{w^*} \mu \qquad\iff\qquad \langle f, \mu_n\rangle \to \langle f, \mu\rangle \quad \forall\, f \in C(X).$$

Strong convergence (i.e., convergence in norm) implies weak-* convergence:

$$\|\mu_n - \mu\| \to 0 \implies \mu_n \xrightarrow{w^*} \mu.$$

A sequence $\{\mu_n\}_{n=1}^{\infty} \subset M(X)$ is norm-bounded if $\sup_{n\in\mathbb{N}} \|\mu_n\| < \infty$. The unit ball in $M(X)$ is weak-* compact (Alaoglu's Theorem), and consequently every norm-bounded sequence of signed measures has weak-* limit points in $M(X)$.

The unit ball in $M(X)$ is weak-* metrizable owing to the separability of $C(X)$. Let $\{f_j\}_{j=1}^{\infty}$ be a dense subset of $C(X)$. Then weak-* convergence need only be checked with respect to the elements of $\{f_j\}_{j=1}^{\infty}$. In particular,

$$\mu_n \xrightarrow{w^*} \mu \quad\iff\quad \langle f_j, \mu_n\rangle \to \langle f_j, \mu\rangle \quad \forall\, j \in \mathbb{N}.$$

Here is a metric compatible with the weak-* topology:

$$d(\mu_1, \mu_2) = \sum_{j=1}^{\infty} 2^{-j} \wedge |\langle f_j, \mu_1\rangle - \langle f_j, \mu_2\rangle|.$$

Given a norm-bounded sequence $\{\mu_n\}_{n=1}^{\infty}$,

$$\mu_n \xrightarrow{w^*} \mu \quad\iff\quad d(\mu_n, \mu) \to 0.$$

[6]The lattice operations on $M(X)$ are given by

$$\mu \vee \nu(A) = \sup\{\mu(B) + \nu(A \setminus B) : B \in \mathcal{B}(X) \text{ and } B \subset A\}$$
$$\mu \wedge \nu(A) = \inf\{\mu(B) + \nu(A \setminus B) : B \in \mathcal{B}(X) \text{ and } B \subset A\}.$$

[7]Moreover, $M(X)$ is an AL-space. An AL-space is a norm complete L-space, which is a normed Riesz space with an L-norm. An L-norm satisfies the following condition: $x, y \geqslant 0$ implies $\|x+y\| = \|x\| + \|y\|$.

[8]The dual pair $\langle C(X), M(X)\rangle$ is a Riesz pair (but not a symmetric Riesz pair). As such, a positive vector $f \in C(X)^+$ is strictly positive, written $f \gg 0$, if f acts as a strictly positive linear functional on $M(X)$ when considered as a member of $M(X)^\sim$, the order dual of $M(X)$. This requires $f(x) > 0$ for all $x \in X$.

Given $A \subset M(X)$, the *annihilator* of A is defined by

$$A^{\perp} := \{f \in C(X) : \langle f, \mu \rangle = 0 \quad \forall\, \mu \in A\}. \tag{1.2}$$

Theorem 5.96 in Aliprantis and Border (1999) asserts that if A is a subspace of $M(X)$, then the following are equivalent: (i) $A^{\perp} = \{\mathbf{0}\}$, (ii) A separates the points of $C(X)$, and (iii) A is weak-* dense in $M(X)$.

Let δ_x denote the point mass located at $x \in X$, where

$$\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \qquad \forall\, A \in \mathcal{B}(X).$$

Note $\|\delta_x\| = |\delta_x|(X) = \delta_x(X) = 1$. Let $\mathbf{1}_A$ denote the characteristic function for $A \in \mathcal{B}(X)$ where

$$\mathbf{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \qquad \forall\, x \in X.$$

Let $\mathbf{1} := \mathbf{1}_X$ and $\mathbf{0} := \mathbf{1}_{\varnothing}$.

Since X is compact metrizable, it is separable. Therefore, there is a countable dense set $\{x_n\}_{n=1}^{\infty} \subset X$. Since $(\{\delta_{x_n}\}_{n=1}^{\infty})^{\perp} = \{\mathbf{0}\}$, $\{\delta_{x_n}\}_{n=1}^{\infty}$ is dense in $M(X)$. Thus finite linear combinations of $\{\delta_{x_n}\}_{n=1}^{\infty}$ can be used to approximate any element of $M(X)$ arbitrarily well. For example, $\mathbb{N}$ is dense in $\mathbb{N}_{\infty}$ (its one-point compactification). Therefore, the set $\{\delta_x\}_{x \in \mathbb{N}}$ can be used to approximate δ_{∞} arbitrarily well. As another example, let $\mathbb{Q}_{[0,1]}$ denote the rationals in $[0,1]$. Since $\mathbb{Q}_{[0,1]}$ is dense in $[0,1]$, point masses on $\mathbb{Q}_{[0,1]}$ can be used to approximate Lebesgue measure on $[0,1]$ arbitrarily well.

Order structure. The positive cone of $M(X)$ is

$$M(X)^{+} = \{\mu \in M(X) : \mu(A) \geqslant 0 \;\forall\, A \in \mathcal{B}(X)\}.$$

The strongly positive cone is $M(X)^{++} = M(X) \setminus \{0\}$. The positive cone of $C(X)$ is

$$C(X)^{+} = \{f \in C(X) : \langle f, \mu \rangle \geqslant 0 \;\mu \in M(X)^{+}\}$$

and its strongly positive cone is $C(X)^{++} = C(X)^{+} \setminus \{0\}$. We say $\mu \in M(X)^{++}$ is strictly positive if $\langle f, \mu \rangle > 0$ for all $f \in C(X)^{++}$ and $f \in C(X)^{++}$ is strictly positive if $\langle f, \mu \rangle > 0$ for all $\mu \in M(X)^{++}$.

Numeraire measure. Fix a positive measure $\lambda \in M(X)$. By the Lebesgue decomposition, every signed measure $\mu \in M(X)$ has a unique representation $\mu = \mu^a + \mu^s$ where $\mu^a \ll \lambda$ (i.e., μ^a is absolutely continuous with respect to λ) and $\mu^s \perp \lambda$ (i.e., μ^s and λ are mutually singular).[9] In fact, λ induces a decomposition of $M(X)$ into the direct sum of projection bands: $B_{\lambda} \oplus B_{\lambda}^{\mathrm{d}} = M(X)$, where

$$B_{\lambda} = \{\mu \in M(X) : \mu \ll \lambda\} \qquad \text{and} \qquad B_{\lambda}^{\mathrm{d}} = \{\mu \in M(X) : \mu \perp \lambda\}.$$

[9] In a Riesz space each pair of vectors has a supremum and and infimum: $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$. In particular, $x^{+} = x \vee 0$, $x^{-} = (-x) \vee 0$; in addition, $|x| = x^{+} + x^{-}$. Vectors x and y are *disjoint* or *orthogonal*, written $x \perp y$, if $|x| \wedge |y| = 0$. From Lemma 9.56 in Aliprantis and Border (1999) we have the following: Given $\mu, \nu \in M(X)$, $\mu \perp \nu$ if and only if there exists some $A \in \mathcal{B}(X)$ such that $|\mu|(A) = |\nu|(A^c) = 0$.

The bands B_λ and B_λ^{d} are both linear subspaces of $M(X)$ and they are disjoint complements: If $\mu \in B_\lambda$ and $\nu \in B_\lambda^{\mathrm{d}}$, then $\mu \perp \nu$.

Define $L_1(\lambda) := L_1(X, \mathcal{B}(X), \lambda)$. The norm of $z \in L_1(\lambda)$ is $\|z\|_1^\lambda = \int_X |z| \, d\lambda$. Let $\| \, \|_1^\lambda$ denote the norm topology. Note $C(X) \subset L_1(\lambda)$, since $\|f\|_1^\lambda = \int_X |f| \, d\lambda < \infty$. Consider the positive linear operator $T_\lambda : B_\lambda \to L_1(\lambda)$, where

$$T_\lambda(\mu) = d\mu/d\lambda. \tag{1.3}$$

Note $\|T_\lambda(\mu)\|_1^\lambda = \|\mu\|$. The inverse operator $T_\lambda^{-1} : L_1(\lambda) \to B_\lambda$ is defined by $T_\lambda^{-1}(z)(B) = \int_B z \, d\lambda$ for all $B \in \mathcal{B}(X)$. Since T_λ is a lattice isometry, the Banach lattices B_λ and $L_1(\lambda)$ are identical from the point of view of Riesz spaces.

The *support* of a positive measure λ (if it exists[10]) is a closed set (denoted $\operatorname{supp} \lambda$) satisfying (i) $\lambda\big((\operatorname{supp} \lambda)^c\big) = 0$ and (ii) if A is open and $A \cap \operatorname{supp} \lambda \neq \varnothing$, then $\lambda(A \cap \operatorname{supp} \lambda) > 0$. For our purposes, what is important is this: If $\operatorname{supp} \lambda = X$, then $B_\lambda^\perp = \{\mathbf{0}\}$ and therefore B_λ is weak-* dense in $M(X)$. If B_λ is dense in $M(X)$, then any element of $M(X)$ can be approximated arbitrarily well by elements in B_λ, all of which have densities with respect to λ. Consequently, we will call λ a *numeraire measure* if $0 \leqslant \lambda \in M(X)$ and $\operatorname{supp} \lambda = X$. It follows that if λ is a numeriare measure, then $\overline{B}_\lambda = M(X)$ in the weak-* topology. [Need to show that any weak-* continuous linear functional on B_λ can be extended to $M(X)$.]

Natural measure. Fix a positive measure $\varphi \in B_\lambda$ for which $\lambda \ll \varphi$. In other words, φ is equivalent to λ. Since φ and λ are equivalent, $B_\varphi = B_\lambda$. Then $\xi := T_\lambda(\varphi) = d\varphi/d\lambda$ is the density of the payout φ in terms of the numeraire measure. Define $\pi := T_\varphi(\lambda) = d\lambda/d\varphi = \xi^{-1}$. Note $T_\lambda(\mu) \, T_\varphi(\lambda) = (d\mu/d\lambda) \, (d\lambda/d\varphi) = d\mu/d\varphi = T_\varphi(\mu)$. Then for $\mu \in B_\lambda$,

$$\langle \mathbf{1}, \mu \rangle = \int_X z \, \pi \, d\varphi,$$

where $z = T_\lambda(\mu)$. If $\varphi(X) = 1$, then $\int_X z \, \pi \, d\varphi = E^\varphi[z \, \pi]$. If, in addition, φ is the *natural measure* (also known as the *physical measure*), then we say π is the state-price deflator.

Reference measure. Define $L_1(\eta) := L_1(X, \mathcal{B}(X), \eta)$ where η is a positive σ-finite Borel measure η such that $\lambda \ll \eta$. (Note $\eta(X) < \infty \implies \eta \in M(X)$.) We refer to η as the *reference measure*. (In our examples, η is either the counting measure or the Borel measure on the real line.) We refer to $G := d\lambda/d\eta = T_\eta(\lambda) \in L_1(\eta)$ as the *pricing function*. Note $\lambda = T_\eta^{-1}(G) \in M(X)$. In addition, $z \in L_1(\lambda) \iff z \, G \in L_1(\eta)$ and $\int_X z \, d\lambda = \int_X z \, G \, d\eta$. Given η, choosing $G \in L_1(\eta)$ is equivalent to choosing $\lambda \in M(X)$.

Finally, let $F = d\varphi/d\eta$. Then

$$E^\varphi[z \, \pi] = \int_X z \, \pi \, F \, d\eta = \int_X \left(\frac{d\mu}{d\lambda}\right) \left(\frac{d\lambda}{d\varphi}\right) \left(\frac{d\varphi}{d\eta}\right) d\eta = \int_X d\mu,$$

where μ, λ, φ, and η are all Borel measures on $\mathcal{B}(X)$ and (with the possible exception of η) all are in $M(X)$.

[10]Given the separability of X, the support of $0 \leq \lambda \in M(X)$ exists.

Convergence in $L_1(\lambda)$ and uniform integrability. Given a sequence $\{z_n\}_{n=1}^{\infty}$ where $z_n \in L_1(\lambda)$ and $z \in L_1(\lambda)$. Consider the following modes of convergence. First, convergence in the L_1 norm:

$$z_n \xrightarrow{\| \|_1^\lambda} z \iff \|z_n - z\|_1^\lambda \to 0.$$

We can restate the norm convergence of functions in terms of the norm convergence of measures: Given a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset B_\lambda$ and $\mu \in B_\lambda$,

$$\|\mu_n - \mu\| \to 0 \iff \|T_\lambda(\mu_n - \mu)\|_1^\lambda \to 0.$$

Second, convergence λ-almost everywhere:

$$z_n \xrightarrow{\lambda\text{-a.e.}} z \qquad \iff \qquad z_n(x) - z(x) \to 0 \quad \forall\, x \in X \setminus E \quad \text{where } \lambda(E) = 0.$$

Third, convergence in λ-measure:

$$z_n \xrightarrow{\lambda} z \qquad \iff \qquad \lambda(\{x \in X : |z_n(x) - z(x)| > \varepsilon\}) \to 0 \quad \forall\, \varepsilon > 0.$$

Relations among the modes of convergence:

$$z_n \xrightarrow{\lambda\text{-a.e.}} z \implies z_n \xrightarrow{\lambda} z$$

and

$$z_n \xrightarrow{\| \|_1^\lambda} z \implies (i)\ z_n \xrightarrow{\lambda} z \text{ and } (ii)\ z \text{ is the unique } \lambda\text{-a.e. limit point.}$$

Theorem 1. Fix $0 \leqslant \lambda \in M(X)$. Given $\{\mu_n\}_{n=1}^{\infty} \subset B_\lambda$ and $\mu = \mu^a + \mu^s$, where $\mu^a \in B_\lambda$ and $\mu^s \in B_\lambda^{\mathrm{d}}$. Then $\mu_n \xrightarrow{w^*} \mu \implies T_\lambda(\mu_n) \xrightarrow{\lambda} T_\lambda(\mu^a)$.

Proof. Christian will supply proof. □

The sequence $\{z_n\}_{n\in\mathbb{N}} \subset L_1(\lambda)$ is uniformly integrable (UI) if

$$\lim_{\alpha\to\infty} \sup_{n\in\mathbb{N}} \int_{|z_n|\geq\alpha} |z_n|\, d\lambda = 0.$$

Theorem 2. Fix $0 \leqslant \lambda \in M(X)$. Given the sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset B_\lambda$, the sequence $\{T_\lambda(\mu_n)\}_{n\in\mathbb{N}}$ is UI if and only if

$$\sup_{n\in\mathbb{N}} \|\mu_n\| < \infty \qquad \text{and} \qquad \lim_{\lambda(A)\to 0} \sup_{n\in\mathbb{N}} |\mu_n|(A) = 0.$$

Doob (1994) describes the second condition as *uniform absolute continuity* of the sequence of measures $\{\mu_n\}_{n\in\mathbb{N}}$.

Proof. See Doob (1994, p. 94). □

Theorem 3. Fix $0 \leqslant \lambda \in M(X)$. Given $\{z_n\}_{n=1}^{\infty} \subset L_1(\lambda)$. If $z_n \xrightarrow{\lambda} z$, then $\|z_n - z\|_1^\lambda \to 0$ if and only if $\{z_n\}_{n=1}^{\infty}$ is UI.

Proof. See Doob (1994, p. 95). □

Theorem 4 (Dunford–Pettis)**.** A subset of $L_1(\lambda)$ is UI if and only if it is relatively weakly compact.

The following theorem is implied by the Dunford–Pettis Theorem.

Theorem 5. Fix $0 \leqslant \lambda \in M(X)$. Given $S := \{\mu_n\}_{n=1}^{\infty} \subset B_\lambda$. Let $\overline{S}$ denote the weak-* closure of S in $M(X)$. Then $\{T_\lambda(\mu_n)\}_{n=1}^{\infty}$ is UI if and only if $\overline{S} \subset B_\lambda$.

Proof. To be supplied. ☐

Corollary 1. Fix $0 \leqslant \lambda \in M(X)$. Given $\{\mu_n\}_{n=1}^{\infty} \subset B_\lambda$ and $\mu \in B_\lambda$. Then $\mu_n \xrightarrow{w^*} \mu \iff \mu_n \xrightarrow{\|\,\|} \mu$.

In other words, $\mu_n \xrightarrow{w^*} \mu \iff T_\lambda(\mu_n) \xrightarrow{\|\,\|_1^\lambda} T_\lambda(\mu)$.

Filtration and stochastic processes. In this section, we use a filtration to generate a sequence in $L_1(\lambda)$ from a signed measure $\mu \in M(X)$.

Let $\{\mathcal{F}_i\}_{i=1}^{\infty}$ be a filtration, where $\mathcal{F}_1 = \{\varnothing, X\}$, $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all $n \in \mathbb{N}$, and $\mathcal{F}_\infty = \sigma(\cup_{i=1}^{\infty} \mathcal{F}_i)$. In addition, assume $\mathcal{F}_\infty = \mathcal{B}(X)$. Fix a σ-finite measure space $(X, \mathcal{B}(X), \lambda)$. Then $(X, \mathcal{B}(X), \{\mathcal{F}_i\}_{i=1}^{\infty}, \lambda)$ is a filtered measure space.

Given $\mu \in M(X)$, form the Lesbegue decomposition with respect to λ: $\mu = \mu^a + \mu^s$ and define $z := d\mu^a/d\lambda$. Define the restrictions of μ and λ to $\mathcal{F}_n$:

$$\hat{\mu}_n(A) = \mu(A) \quad \text{and} \quad \lambda_n(A) = \lambda(A) \quad \forall\, A \in \mathcal{F}_n.$$

Now suppose $\hat{\mu}_n \ll \lambda_n$ for all $n \in \mathbb{N}$ (i.e., the λ-null sets not shared by μ are only in $\mathcal{B}(X)$ and not in any $\mathcal{F}_n$). Define $z_n := d\hat{\mu}_n/d\lambda_n$. Note $\{z_n\}_{n=1}^{\infty}$ is adapted to the filtration. Define the 'conditional' measure μ_n via[11]

$$\mu_n(A) := T_\lambda^{-1}(z_n).$$

It can be shown that $\mu_n \xrightarrow{w^*} \mu$. In other words,

$$T_\lambda^{-1}(z_n) \xrightarrow{w^*} \mu.$$

Also note, $\|z_n\| = \int_X |z_n|\, d\lambda = \int_X |z_n|\, d\lambda_n = \|\hat{\mu}_n\| = \|\mu\|$. Thus $\{z_n\}_{n=1}^{\infty}$ is norm bounded. Moreover, $\int_X z_n\, d\lambda_n = \hat{\mu}_n(X) = \mu(X)$ for all $n \in \mathbb{N}$. It can be shown that $z_n \xrightarrow{\lambda\text{-a.e.}} z$. Note, however, $\int_X z\, d\lambda = \mu(X) - \mu^s(X) \neq \mu(X)$ unless $\mu^s(X) = 0$.

Let φ_n denote the restriction of φ to $\mathcal{F}_n$. Note $\varphi_n(X) = \varphi(X)$. Define $\pi_n = d\lambda_n/d\varphi_n$. Note $\{\pi_n\}_{n=1}^{\infty}$ is adapted to the filtration. If $\varphi(X) = 1$, then $\{z_n\, \pi_n\}_{n=1}^{\infty}$ is a φ-martingale. In addition, $z_n \xrightarrow{\varphi\text{-a.e.}} z$.

2. Arbitrage

Our treatment of arbitrage follows Kreps (1981) and Clark (1993, 2000, 2002).[12] Fix a topological space of payouts (S, τ) and a cone K. In this paper, S is a Banach lattice and $K = S^+ = \{x \in S : x \geqslant 0\}$, the positive cone of S. Let $S^{++} = S^+ \setminus \{0\}$ denote the strongly positive cone (i.e., the positive cone with the origin deleted).[13]

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[11] In shorthand notation we have $d\mu_n = \frac{d\hat{\mu}_n}{d\lambda_n}\, d\lambda$.

[12] Which see for omitted details.

[13] For any subspace E in S, let $E^+ := E \cap S^+$ denote the positive cone of E and let $E^{++} := E \cap S^{++}$ denote the strongly positive cone of E. The topological dual vector space E^* consists of all continuous linear functionals on E. Its positive cone is given by

$$(E^*)^+ := \{p \in S^* : p(x) \geqslant 0\ \forall\, x \in E^+\},$$

Let $\mathbb{M} \subset S$ denote the space of marketed claims and let $\overline{\mathbb{M}}$ denote the τ-closure of $\mathbb{M}$. The prices of marketed contingent claims are given by a linear functional $\pi : \mathbb{M} \to \mathbb{R}$, which embodies the law of one price. We assume there exists $m_0 \in \mathbb{M}$ such that m_0 is strictly positive (relative to S) and that $\pi(m_0) > 0$. For some purposes it may be convenient to posit a set of marketed securities $\mathbb{M}_0$ and to define the space of marketed claims as $\mathbb{M} = \mathbf{sp}(\mathbb{M}_0)$, where $\mathbf{sp}(\mathbb{M}_0)$ is the linear span of $\mathbb{M}_0$.

A *valuation operator* V is a continuous strictly positive linear functional on S that extends π; i.e., $V(m) = \pi(m)$ for every $m \in \mathbb{M}$. Let $\mathcal{P}$ denote the collection of all positive linear extensions $p : S \to \mathbb{R}$ of π. We say that a contingent claim $x \in S$ is *priced by arbitrage* whenever $p(x)$ has the same value for every $p \in \mathcal{P}$.

Define the feasible set $F := \{m \in \mathbb{M} : \pi(m) = 0\}$. Let $\overline{F}$ denote the closure of F. An *arbitrage* is an element of $S^{++} \cap F$. An *approximate arbitrage* is an element of $S^{++} \cap \overline{F}$. We can characterize the absence of arbitrage opportunities and the absence of approximate arbitrage opportunities as follows:

$$S^{++} \cap F = \varnothing \tag{NA}$$

and

$$S^{++} \cap \overline{F} = \varnothing. \tag{NAA}$$

(NA) holds if and only if π is strictly positive. It is of some interest to note that when $\mathbb{M}$ is closed, $F = \overline{F}$ and consequently (NAA) is equivalent to (NA). In particular, if $\mathbb{M} = S$, then π is a valuation operator.[14]

The following theorem combines Theorems 1–3 from Clark (2002).

Theorem 6. Suppose S is a separable Banach lattice such that the norm is order-continuous and $\overline{\mathbb{M}}$ is a sublattice of S.

(1) Then there exists a valuation operator if and only if (NAA) holds.
(2) Then there exists a unique valuation operator if and only if (NAA) holds and $\overline{\mathbb{M}} = S$.
(3) If (NAA) holds, then a contingent claim $x \in S$ is priced by arbitrage if and only if $x \in \overline{\mathbb{M}}$.

Clark notes the theorem applies to inseparable L_p spaces (for $1 \leqslant p < \infty$), so separability per se is not essential.

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and its strongly positive cone $(E^*)^{++} := \{p \in (E^*)^+ : p \neq 0\}$. We say a vector $x \in E^{++}$ is strictly positive (relative to E) provided that $p(x) > 0$ for every $p \in (E^*)^{++}$, and we say that a linear functional $p \in (E^*)^{++}$ is strictly positive if provided that $p(x) > 0$ for every $x \in E^{++}$.

Letting $(S, \tau) = (M(X), w^*)$, we have

$$\begin{aligned} M(X)^+ &= \{\mu \in M(X) : \mu(A) \geqslant 0 \ \forall\, A \in \mathcal{B}(X)\} \\ C(X)^+ &= \{f \in C(X) : \langle f, \mu\rangle \geqslant 0 \ \forall\, \mu \in M(X)^+\}. \end{aligned}$$

[14] For comparison, consider

$$S^{++} \cap \widehat{F} = \varnothing, \tag{NFL}$$

where $\widehat{F} := (\overline{S^+ - F}) \cap (\overline{F - S^+})$. Clark (2002, Lemma A) proves that (NFL) holds if and only if there are no free lunches in the sense of Kreps (1981). In addition, Clark (2002, Lemma B) proves the following: Suppose S is a Banach lattice such that the norm is order continuous and $\overline{\mathbb{M}}$ is a sublattice of S. If (NAA) holds, then $\widehat{F} = \overline{F}$.

Since every AL-space has an order-continuous norm,[15] the preceding theorem applies to $(M(X), \| \|)$. I suspect the theorem also applies to $(M(X), w^*)$. The assumption of an order-continuous norm is equivalent to the assumption that the norm topology is order-continuous. Therefore, what is (probably) required in our setting is that the weak-* topology is order-continuous (which I'm guessing is true because norm convergence implies weak-* convergence).

Spaces of payouts. Given $0 \leqslant \lambda \in M(X)$ where $\operatorname{supp} \lambda = X$. The following are equivalent:

(1) $z \mapsto \int_X z \, d\lambda$ is a valuation operator on $(L_1(\lambda), \| \|_1^\lambda)$.
(2) $\mu \mapsto \int_X d\mu = \langle \mathbf{1}, \mu \rangle$ is a valuation operator on $(B_\lambda, \| \|)$.
(3) $\mu \mapsto \int_X d\mu = \langle \mathbf{1}, \mu \rangle$ is a valuation operator on $(M(X), w^*)$.

Anatomy of an apparent arbitrage. Let the space of payouts be $(M(X), w^*)$ and let the valuation operator be given by $\mu \mapsto \langle \mathbf{1}, \mu \rangle$. Fix a positive measure $\lambda \in M(X)$ for which $\operatorname{supp} \lambda = X$. Take as given a sequence $\{\mu_n\}_{n=1}^\infty \subset B_\lambda$ where $\mu_n(X) = 0$ for all $n \in \mathbb{N}$ and for which $\mu_n \xrightarrow{w^*} \mu = \mu^a + \mu^s$, where $\mu^a \in B_\lambda$ and $\mu^s \in B_\lambda^{\mathrm{d}}$. By Theorem 1, $T_\lambda(\mu_n) \xrightarrow{\lambda} T_\lambda(\mu^a)$. It follows from weak-* convergence that $\mu(X) = 0$ and thus $\int_X T_\lambda(\mu^a) \, d\lambda = -\mu^s(X)$. If $\mu^s(X) < 0$ we have an apparent arbitrage opportunity, while if $\mu^s(X) > 0$ we have an apparent suicide strategy.[16]

3. Intervals on the real line

In this section we illustrate weak* convergence and apparent arbitrages in the spaces $X = [0, a]$ where $a \in (0, \infty]$. In all cases, the reference measure η is the Borel measure, denoted Bor.[17]

Example 3.1. This example shows that a sequence of 'lump-sum' payouts can converge to a 'lump-sum' payout. Let $X = [0, 1]$. Given a sequence of distinct points $\{x_n\}_{n=1}^\infty \subset [0, 1] \setminus \{x\}$ such that $x_n \to x \in [0, 1]$. The sequence of point masses $\{\delta_{x_n}\}_{n=1}^\infty$ does not converge in norm, since $\|\delta_{x_n} - \delta_x\| = 2$ for all $n \in \mathbb{N}$. Nevertheless, the sequence does converge in the weak* topology:

$$\langle f, \delta_{x_n} \rangle = f(x_n) \underset{n}{\longrightarrow} f(x) = \langle f, \delta_x \rangle \qquad \forall\, f \in C([0, 1]).$$

Example 3.2. This example illustrates how 'lump-sum' payouts can be used to approximate a 'flow' payout. Let $X = [0, 1]$ and let $\lambda = \text{Bor}$. For each $n \in \mathbb{N}$, partition $[0, 1]$ into n intervals of equal length. Define

$$\hat{\lambda}_n := \frac{1}{n} \sum_{i=1}^n \delta_{\frac{2\,i-1}{2\,n}}.$$

[15]Aliprantis and Border (1999, p. 313).
[16]Note, if $\mu^s \neq 0$ (even if $\mu^s(X) = 0$), then $\|\mu_n - \mu\| \not\to 0$.
[17]Lebesgue measure is the completion of the Borel measure (which is defined on the Borel σ-algebra).

Note $\hat{\lambda}_n \xrightarrow{w^*} \lambda$ since

$$\int_{[0,1]} f \, d\hat{\lambda}_n = \frac{1}{n} \sum_{i=0}^{n} f\left(\frac{2\,i-1}{2\,n}\right) \underset{n}{\rightarrow} \int_0^1 f(x)\, dx = \langle f, \lambda \rangle \qquad \forall\, f \in C([0,1]).$$

Example 3.3. This example illustrates an apparent arbitrage. Let $X = [0,1]$ and let $\lambda = \text{Bor}$. Let $\mu = \lambda - \delta_0$, so that $\mu(X) = 0$ and $\|\mu\| = 2$. In addition, $\mu^a = \lambda$, $\mu^s = -\delta_0$, $z = d\mu^a/d\lambda = \mathbf{1}$ (λ-a.e.), and $\int_X z\, d\lambda = 1$.

Define μ_n via $\mu_n(A) = \int_A z_n\, d\lambda$ for all $A \in \mathcal{B}(X)$, where

$$z_n(x) := \begin{cases} 1 - 2^{n-1} & x \in [0, 2^{1-n}] \\ 1 & x \in (2^{1-n}, 1]. \end{cases}$$

(See Figure 1.) Note $\mu_n \xrightarrow{w^*} \mu$ and $z_n \xrightarrow{\lambda\text{-a.e.}} z$. Moreover, $\mu_n(X) = \int_X z_n\, d\lambda = 0$ for all $n \in \mathbb{N}$. Since $\int_X z\, d\lambda = 1$, there is an apparent arbitrage.

Consider the image measure (i.e., the distribution function)

$$F_n(x) = \lambda(\{y \in [0,1] : z_n(y) \le x\}) = \begin{cases} 0 & x < 1 - 2^{n-1} \\ 2^{1-n} & 1 - 2^{n-1} \le x < 1 \\ 1 & 1 \le x. \end{cases}$$

(See Figure 1.) Note

$$\int_{\mathbb{R}} f\, dF_n = 2^{1-n}\, f(1 - 2^{n-1}) + (1 - 2^{1-n})\, f(1) \qquad \forall\, f \in C_b(\mathbb{R}),$$

where $C_b(\mathbb{R})$ is the space of continuous bounded functions on $\mathbb{R}$. Thus,

$$\int_{\mathbb{R}} f\, dF_n \to f(1) = \int_{\mathbb{R}} f\, dF \qquad \forall\, f \in C_b(\mathbb{R}),$$

where

$$F(x) = \begin{cases} 0 & x < 1 \\ 1 & x \ge 1. \end{cases}$$

Therefore, the sequence of image measures converges in distribution to 1. This simply reflects convergence in measure and does not contradict the weak* convergence of of $\{T_\lambda(z_n)\}_{n=1}^{\infty}$.

Example 3.4. (Miller–Modigliani). This example is adapted from the "zero"-dividend example in Gilles and LeRoy (1997).

Consider a firm that starts with one unit of capital that generates earnings at rate $r > 0$. Suppose the firm pays out a constant fraction $\gamma \in (0,1)$ of earnings as dividends. Let $z_\gamma(t)$ denote the rate of flow of dividends at time t and let $s(t)$ denote firm's stock of capital at time t. Then $z_\gamma(t) = \gamma\, r\, s(t)$. The firm acquires additional capital with its retained earnings, and so its capital evolves according to the ordinary differential equation (ODE) $s'(t) = (1-\gamma)\, r\, s(t)$ subject to $s(0) = 1$. The solution to the ODE is $s(t) = e^{(1-\gamma)\, r\, t}$ and therefore $z_\gamma(t) = \gamma\, r\, e^{(1-\gamma)\, r\, t}$.

Let $X = [0, \infty]$ and let $\eta = \text{Bor}$. Let $G(t) = e^{-r\, t}$ be the pricing function, so the numeraire measure is characterized by $\lambda(dt) = G(t)\, \eta(dt) = G(t)\, dt$. Note

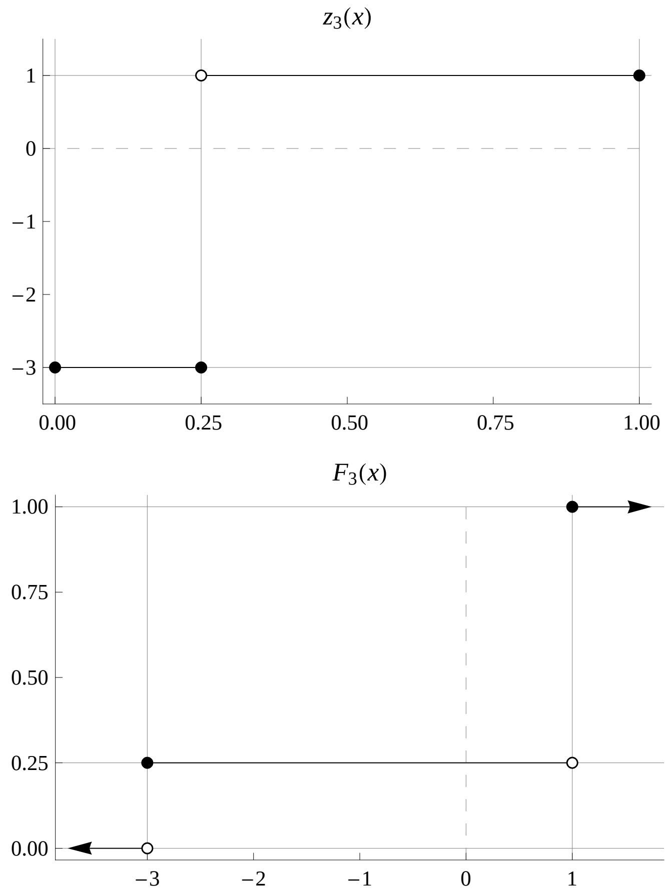

FIGURE 1. $z_3(x)$ and $F_3(x)$.

$\lambda(X) = \int_0^\infty e^{-r\,t}\,dt = 1/r$. Note $z_\gamma \in L_1(\lambda)$ since $\int_X |z_\gamma(t)|\,\lambda(dt) < \infty$. Define $\mu_\gamma := T_\lambda^{-1}(z_\gamma)$. The density of μ_γ with respect to Lebesgue measure is given by $w_\gamma(t) := \gamma\, r\, e^{-\gamma\, r\, t} = z_\gamma(t)\, G(t)$. The value of the firm is independent of the dividend–payout ratio:

$$\mu_\gamma(X) = \int_0^\infty \mu_\gamma(dt) = \int_0^\infty \gamma\, r\, e^{-\gamma\, r\, t}\, dt = 1.$$

Consider what happens when the dividend–payout ratio goes to zero. First, note that all of the firm's value is attributable to dividends in the tail as the dividend–payout ratio goes to zero:

$$\lim_{\gamma \downarrow 0} \mu_\gamma([T,\infty]) = 1 \qquad \text{for every finite } T.$$

Given a sequence $\{\gamma_n\}_{n=1}^\infty \subset (0,1)$ where $\gamma_n \to 0$, we show that $\mu_{\gamma_n} \xrightarrow{w^*} \delta_\infty$. Note

$$\langle f, \mu_\gamma \rangle = \int_0^\infty f(t)\, d\mu_\gamma(dt) = \int_0^\infty f(t)\, w_\gamma(t)\, dt.$$

Thus,

$$\langle f, \mu_{\gamma_n}\rangle \underset{n}{\to} f(\infty) = \langle f, \delta_\infty\rangle \qquad \forall\, f \in C([0,\infty]),$$

where $f(\infty) := \lim_{t\to\infty} f(t)$. Since $\mu_\gamma \xrightarrow{w^*} \delta_\infty$ and $z_\gamma(t) \xrightarrow{\lambda\text{−a.e.}} 0$, buying shares in the 'zero'-dividend firm is an apparent suicide strategy.

4. Countable state space

In this section, let $X = \mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$.[18] $\mathbb{N}_\infty$ is the one-point compactification of $\mathbb{N}$.[19] Let $\boldsymbol{e}_n$ denote the n-th unit coordinate vector; i.e., the sequence whose n-th term is one and every other term is zero and let $\widetilde{\boldsymbol{e}}_n := \sum_{i=n+1}^\infty \boldsymbol{e}_i$. Note $\widetilde{\boldsymbol{e}}_0 = \mathbf{1}_\mathbb{N}$ and $\widetilde{\boldsymbol{e}}_0 + \boldsymbol{e}_\infty = \mathbf{1} = \mathbf{1}_{\mathbb{N}_\infty}$.

Given a sequence $\{f(i)\}_{i=1}^\infty$, define $\mathrm{Lim}[f] := \lim_{i\to\infty} f(i)$. Note that $f \in C(\mathbb{N}_\infty) \iff \mathrm{Lim}[f]$ exists and is finite and $f(\infty) = \mathrm{Lim}[f]$. In addition, note that $\mu \in M(\mathbb{N}_\infty) \iff \mu = \sum_{i=1}^\infty m_i\,\delta_i + m_\infty\,\delta_\infty$, subject to

$$\|\mu\| = |\mu|(\mathbb{N}_\infty) = \sum_{i\in\mathbb{N}_\infty} |m_i| < \infty.$$

Thus, $\mu(\{i\}) = m_i$ for all $i \in \mathbb{N}_\infty$. The duality for $\langle C(\mathbb{N}_\infty), M(\mathbb{N}_\infty)\rangle$ is[20]

$$\langle f, \mu\rangle = \int_X f\,d\mu = \sum_{i\in\mathbb{N}_\infty} f(i)\,m_i = \sum_{i\in\mathbb{N}} f(i)\,m_i + \mathrm{Lim}[f]\,m_\infty.$$

Define $I(\mu) := \{i \in \mathbb{N}_\infty : \mu(\{i\}) \neq 0\}$. Then $\mu' \ll \mu \iff I(\mu') \subseteq I(\mu)$. If $\mu' \ll \mu$, then

$$\frac{d\mu'}{d\mu} = \sum_{i\in I(\mu)} \left(\frac{m_i'}{m_i}\right) \boldsymbol{e}_i.$$

Let $\lambda := \sum_{i=1}^\infty \beta_i\,\delta_i$, where $\beta_n > 0$ for $n \in \mathbb{N}$ and $\overline{\beta} := \sum_{i=1}^\infty \beta_i < \infty$.[21] Define $\widetilde{\beta}_n := \sum_{n+1}^\infty \beta_i$. Note $\lambda(\mathbb{N}_\infty) = \overline{\beta} < \infty$ and $\lambda(\{\infty\}) = 0$. The support of λ is $\mathbb{N}_\infty$.[22] Define $L_1(\lambda) := L_1(\mathbb{N}_\infty, \mathcal{B}(\mathbb{N}_\infty), \lambda)$.

For any $\mu \in M(\mathbb{N}_\infty)$, we have $\mu = \mu^a + \mu^s$ where $\mu^a = \sum_{i=1}^\infty m_i\,\delta_i \ll \lambda$ and $\mu^s = m_\infty\,\delta_\infty \perp \lambda$. For $\mu \in B_\lambda$, $T_\lambda(\mu) = d\mu/d\lambda = \sum_{i=1}^\infty (m_i/\beta_i)\,\boldsymbol{e}_i \in L_1(\lambda)$.[23] Given $z = \sum_{i=1}^\infty z(i)\,\boldsymbol{e}_i \in L_1(\lambda)$, $T_\lambda^{-1}(z) = \sum_{i=1}^\infty z(i)\,\beta_i\,\delta_i$.[24]

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[18] In Appendix A we show how to obtain $\mathbb{N}_\infty$ as a quotient space of the Cantor space of infinite sequences of zeros and ones.

[19] The open sets of $\mathbb{N}_\infty$ are the open sets of $\mathbb{N}$ in the discrete topology and sets of the form $A \cup \{\infty\}$ where A is an open subset of $\mathbb{N}$ and $\mathbb{N} \setminus A$ is compact. Thus A is an infinite set.

[20] $C(\mathbb{N}_\infty)$ is also known as c and $M(\mathbb{N}_\infty)$ is also known as $\ell_1 \oplus \mathbb{R}$. See Aliprantis and Border (1999, Section 15.4).

[21] The reference measure is the counting measure $\eta = \sum_{i=1}^\infty \delta_i + \delta_\infty$ and (assuming λ is the numeraire measure) the pricing function is $G = d\lambda/d\eta = \sum_{i=1}^\infty \beta_i\,\boldsymbol{e}_i$. Also note, $d\delta_n/d\eta = \boldsymbol{e}_n$.

[22] $\mathbb{N}_\infty$ is closed, its complement $\varnothing$ has λ-measure zero, and $\{\infty\}$ is not open (so the λ-measure of $\mathbb{N}_\infty \cap \{\infty\} = \{\infty\}$ need not be positive).

[23] $T_\lambda(\mu)(\infty)$ is arbitrary since $\lambda(\{\infty\}) = 0$. We have set it to zero for convenience.

[24] $T_\lambda^{-1}(z)$ does not depend on $z(\infty)$, which we have set to zero for convenience.

Fix the natural probability measure $\varphi := \sum_{i=1}^{\infty} \alpha_i\, \delta_i$, where $\alpha_i > 0$ for all $n \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \alpha_i = 1$. Let $\widetilde{\alpha}_n := \sum_{i=n+1}^{\infty} \alpha_i$. Note φ is equivalent to λ ($\varphi \ll \lambda$ and $\lambda \ll \varphi$). In particular, $B_\varphi = B_\lambda$, the φ-null sets are $\varnothing$ and $\{\infty\}$, and $\operatorname{supp}\varphi = \mathbb{N}_\infty$. Note $(\mathbb{N}_\infty, \mathcal{B}(\mathbb{N}_\infty), \varphi)$ is a probability space. Define $L_1(\varphi) := L_1(\mathbb{N}_\infty, \mathcal{B}(\mathbb{N}_\infty), \varphi)$. For $\mu \in B_\varphi$, $T_\varphi(\mu) = d\mu/d\varphi = \sum_{i=1}^{\infty} (m_i/\alpha_i)\, \boldsymbol{e}_i$. Let $\pi = T_\varphi(\lambda) = d\lambda/d\varphi = \sum_{i=1}^{\infty} (\beta_i/\alpha_i)\, \boldsymbol{e}_i$. For $z \in L_1(\varphi)$, $T_\varphi^{-1}(z) = \sum_{i=1}^{\infty} \alpha_i\, z(i)\, \delta_i$ and $E^\varphi[z] = \sum_{i=1}^{\infty} z(i)\, \alpha_i$. Note $E^\varphi[\pi] = \sum_{i=1}^{\infty} \pi(i)\, \alpha_i = \overline{\beta}$.

It is convenient to define

$$\widetilde{\delta}_n := \widetilde{\beta}_n^{-1} \sum_{i=n+1}^{\infty} \beta_i\, \delta_i. \tag{4.1}$$

Note $\|\widetilde{\delta}_n\| = \widetilde{\delta}_n(\mathbb{N}_\infty) = \widetilde{\beta}_n^{-1} \sum_{i=n+1}^{\infty} \beta_i = 1$.

Example 4.1. We have $\delta_n \xrightarrow{w^*} \delta_\infty$ since

$$\langle f, \delta_n \rangle = f(n) \underset{n}{\rightarrow} \operatorname{Lim}[f] = \langle f, \delta_\infty \rangle \qquad \forall\, f \in C(\mathbb{N}_\infty).$$

Remarks. The density of δ_n with respect to λ is $T_\lambda(\delta_n) = \beta_n^{-1}\, \boldsymbol{e}_n$ for $n \in \mathbb{N}$. Note $\lim_{n\to\infty} \beta_n^{-1} = \infty$. Since $\delta_\infty \in B_\lambda^{\mathrm{d}}$, $\{T_\lambda(\delta_n)\}_{n=1}^{\infty}$ is not uniformly integrable. Also note $T_\lambda(\delta_n) \xrightarrow{\lambda\text{-a.e.}} \mathbf{0}$.

Interpretation. The very-long discount (VLD) bond is the weak-* limit of a sequence of zero-coupon bonds. (This example is adapted from Gilles and LeRoy (1997).)

We interpret β_n as the discount factor: $\beta_n = (1+r)^{-n}$, where $r > 0$ is the interest rate.[25] The payout to a zero-coupon bond that pays one unit when it matures at time n is $\boldsymbol{e}_n$ and its value is β_n. The payout to a zero-coupon bond that pays $(1+r)^n$ when it matures at time n is $\beta_n^{-1}\, \boldsymbol{e}_n$ and its value is 1. We can identify the payout to this bond with the measure $\delta_n = T_\lambda^{-1}(\beta_n^{-1}\, \boldsymbol{e}_n)$. The payout to the VLD bond is δ_∞ and its value is $\delta_\infty(\mathbb{N}_\infty) = 1$.

The payout to the VLD bond can be replicated sequentially as follows. Invest \$1 at time zero in one-period debt and rollover the investment each period. The reverse transaction (a Ponzi scheme of sorts: borrow \$1 at time zero and rollover the outstanding debt each period) is an apparent arbitrage.

Example 4.2. We have $\widetilde{\delta}_n \xrightarrow{w^*} \delta_\infty$ since

$$\langle f, \widetilde{\delta}_n \rangle = \widetilde{\beta}_n^{-1} \sum_{i=n+1}^{\infty} \beta_i\, f(i) \underset{n}{\rightarrow} \operatorname{Lim}[f] = \langle f, \delta_\infty \rangle \qquad \forall\, f \in C(\mathbb{N}_\infty).$$

Remarks. The density of $\widetilde{\delta}_n$ with respect to λ is

$$T_\lambda(\widetilde{\delta}_n) = \widetilde{\beta}_n^{-1}\, \widetilde{\boldsymbol{e}}_n. \tag{4.2}$$

Note $\lim_{n\to\infty} \widetilde{\beta}_n^{-1} = \infty$. Since $\delta_\infty \in B_\lambda^{\mathrm{d}}$, $\{T_\lambda(\delta_n)\}_{n=1}^{\infty}$ is not uniformly integrable. Also note $T_\lambda(\widetilde{\delta}_n) \xrightarrow{\lambda\text{-a.e.}} \mathbf{0}$.

[25]Note $\overline{\beta} = 1/r$. If $r = 0$, then $\lambda(\mathbb{N}_\infty) = \infty$ and λ would be σ-finite but not finite.

Interpretation. Let the state of the world be characterized by the first occurrence of RED on a roulette wheel. The suicide strategy is the weak-* limit of a sequence of normalized fair bets on BLACK.

We interpret β_n as the probability that the first RED occurs on the n-th spin. We assume $\overline{\beta} = 1$. The price of a fair bet that pays one unit if the first RED occurs on the n-th spin equals the probability, β_n.[26] The payout of this bet is $\boldsymbol{e}_n$. It is convenient to normalize fair bets so that their cost is \$1. Thus the payout to a normalized fair bet that the first RED occurs on the n-th spin is $\beta_n^{-1}\,\boldsymbol{e}_n$. (For example, if $\beta_n = 2^{-n}$, then $\beta_n^{-1} = 2^n$.)

The payout to a normalized fair bet that the first n spins are all BLACK is $\widetilde{\beta}_n^{-1}\,\widetilde{\boldsymbol{e}}_n$. (For example, if $\beta_n = 2^{-n}$, then $\widetilde{\beta}_n^{-1} = 2^n$.) We can identify this payout with the measure $\widetilde{\delta}_n = T_\lambda^{-1}(\widetilde{\beta}_n^{-1}\,\widetilde{\boldsymbol{e}}_n)$. The sequence of payouts $\{\widetilde{\delta}_n\}_{n=1}^\infty$ converges to δ_∞, which is the payout to a normalized fair bet that the first RED never occurs. This payout does not have a density with respect to λ; nevertheless, the payout can be approximated arbitrarily well with payouts that do have densities. The bet can be executed sequentially by first betting \$1 on BLACK and continuing to bet any and all winnings on BLACK until RED occurs.

The doubling strategy. The doubling strategy involves betting on RED and doubling the bet each time BLACK occurs. The first bet of \$1 is financed by borrowing, as are all subsequent required bets. When RED occurs, the loans are repaid, after which \$1 remains.

Formally, we can model the doubling strategy as follows: Define $\mu_n := \lambda - \widetilde{\delta}_n$. Note $\mu_n(\mathbb{N}_\infty) = 0$. The payout to the doubling strategy (in terms of the numeraire) after the n-th spin is $T_\lambda(\mu_n) = \mathbf{1}_{\mathbb{N}} - \widetilde{\beta}_n^{-1}\,\widetilde{\boldsymbol{e}}_n$. For example, if $\beta_n = 2^{-n}$, then $T_\lambda(\mu_n) = \sum_{i=1}^n \boldsymbol{e}_i + (1 - 2^n)\,\widetilde{\boldsymbol{e}}_n$. Finally, we have $\mu_n \xrightarrow{w^*} \lambda - \delta_\infty \notin M(X)^{++}$.

Below we present a filtration that allows us to put the suicide and doubling strategies into a fully dynamic setting.

Convergence in distribution of the suicide strategy. Given a sequence of random variables defined on $\mathbb{N}_\infty$, we can examine the corresponding sequence of image measures on the real line and see to what it converges in distribution. We show that the suicide strategy converges in distribution to zero.

The distribution function for $T_\lambda(\widetilde{\delta}_n) = \widetilde{\beta}_n^{-1}\,\widetilde{\boldsymbol{e}}_n$ is

$$F_n(x) := \lambda(\{i : T_\lambda(\widetilde{\delta}_n)(i) \leqslant x\}) = \begin{cases} 0 & x < 0 \\ 1 - \widetilde{\beta}_n & 0 \leqslant x < \widetilde{\beta}_n^{-1} \\ 1 & \widetilde{\beta}_n^{-1} \leqslant x. \end{cases}$$

Note,

$$\int_{\mathbb{R}} f\,dF_n = (1 - \widetilde{\beta}_n)\,f(0) + \widetilde{\beta}_n\,f(\widetilde{\beta}_n^{-1}) \qquad \forall f \in C_b(\mathbb{R}),$$

[26]Here we are assuming $\beta_n = \alpha_n$, where α_n is physical probability and β_n is the equivalent risk-neutral probability.

where $C_b(\mathbb{R})$ is the space of continuous bounded functions on $\mathbb{R}$. Also note,

$$\lim_{n\to\infty} \widetilde{\beta}_n = 0.$$

Therefore,

$$\int_{\mathbb{R}} f\, dF_n \to f(0) = \int_{\mathbb{R}} f\, dF \qquad \forall\, f \in C_b(\mathbb{R}),$$

where

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

is the distribution for a random variable that is identically zero. In other words, we find that the sequence of random variables converges in distribution to the constant zero. This does not contradict our earlier finding that the suicide strategy does not weak* converge to zero. Although both sequences involve weak convergence in the generic sense, the spaces of measures (and of test functions) are different. On the one hand we have $\{\widetilde{\delta}_n\}_{n=1}^{\infty}$, a sequence of meausres on $\mathbb{N}_\infty$, while on the other hand we have $\{F_n\}_{n=1}^{\infty}$, a sequence of distribution functions for Lebesgue–Stieltjes measures on the real line.

Representations. Here we consider a variety of representations for the continuous linear functional $\mu \mapsto \mu(X)$. In particular, is there an equivalent measure representation?

For $\mu \in B_\lambda = B_\varphi$, we have

$$\mu(X) = \int_X d\mu = \int_X \left(\frac{d\mu}{d\lambda}\right) d\lambda = \int_X \left(\frac{d\mu}{d\lambda}\right)\left(\frac{d\lambda}{d\varphi}\right) d\varphi = E^{\varphi}\left[\left(\frac{d\mu}{d\lambda}\right)\pi\right].$$

As long as $B_\varphi \neq M(X)$ (i.e., as long as $B_\lambda^{\mathrm{d}} \neq \varnothing$), there can be no measure that is equivalent to the natural measure φ that allows us to compute all values as an expectation.

In this simple setting where $X = \mathbb{N}_\infty$, it is possible to find a numeraire measure for which there are no nonempty null sets.[27] For example, define

$$\zeta := \lambda + a\,\delta_\infty,$$

where $a > 0$. Note $B_\zeta = M(X)$ and we therefore can identify $M(X)$ with $L_1(\zeta)$. Of course, ζ is not equivalent to either φ or λ. Then for every $\mu \in M(X)$ we can write

$$\mu(X) = \int_X \left(\frac{d\mu}{d\zeta}\right) d\zeta.$$

Now suppose $\overline{\beta} < 1$ and $a = 1 - \overline{\beta}$. In this case, $\zeta = \lambda + (1 - \overline{\beta})\,\delta_\infty =: \xi$ and $\xi(X) = 1$, and we can write

$$\mu(X) = E^{\xi}\left[\left(\frac{d\mu}{d\xi}\right)\right].$$

This is a nonequivalent measure representation. For example, let $\mu = \delta_\infty$. Then $d\mu/d\xi = a^{-1}\,\boldsymbol{e}_\infty \in L_1(\xi)$.

[27] This is not possible in general.

How good an approximation can we obtain? Given any $\mu \in M(X)$, let

$$\mu_n := \sum_{i=1}^{n} m_i\, \delta_i + (\widetilde{m}_n + m_\infty)\, \widetilde{\delta}_n.$$

Note $\mu_n(X) = \mu(X)$ and $\mu_n \xrightarrow{w^*} \mu$. In particular,

$$\xi_n := \sum_{i=1}^{n} \beta_i\, \delta_i + \big(\widetilde{\beta}_n + (1 - \overline{\beta})\big)\, \widetilde{\delta}_n.$$

Treating μ_n and ξ_n as approximations to μ and ξ (respectively), we can write

$$\mu(X) \approx \int_X \left(\frac{d\mu_n}{d\xi_n}\right) d\xi_n = E^{\xi_n}\left[\left(\frac{d\mu_n}{d\xi_n}\right)\right],$$

where ξ_n is an equivalent measure (to φ). The approximation can be made arbitrarily good by letting n get arbitrarily large.

Arbitrage and approximate arbitrage. Here we examine the conditions for no arbitrage and no approximate arbitrage. The marketed securities are the Arrow–Debreu securities (one for each finite state) and a bond that pays one unit in every finite state.

The payout to the n-th A–D security is $\boldsymbol{e}_n$ and the payout to the bond is $\mathbf{1}_{\mathbb{N}} = \sum_{i=1}^{\infty} \boldsymbol{e}_i$. The set of marketed payouts is $\mathbb{M}_0 := \{\mathbf{1}_{\mathbb{N}}\} \cup \{\boldsymbol{e}_n\}_{n=1}^{\infty}$. The space of marketed payouts is $\mathbb{M} := \mathbf{sp}(\mathbb{M}_0)$. Note $\mathbb{M} \subset C(\mathbb{N}_\infty)$. Note $z \in \mathbb{M}$ if and only if $z = a_0\, \mathbf{1}_{\mathbb{N}} + \sum_{i=1}^{\infty} a_i\, \boldsymbol{e}_i$ where the sequence $\{a_i\}_{i=1}^{\infty}$ has finite support. The cost of $z \in \mathbb{M}$ is $V(z) = a_0\, V_0(\mathbf{1}_{\mathbb{N}}) + \sum_{i=1}^{\infty} a_i\, V_0(\boldsymbol{e}_n)$, where $V_0(\mathbf{1}_{\mathbb{N}})$ and $V_0(\boldsymbol{e}_n)$ are the given prices of the marketed securities. Let $V_0(\boldsymbol{e}_n) = \beta_n$ and $V_0(\mathbf{1}_{\mathbb{N}}) = B > 0$.[28] Thus $V(z) = a_0\, B + \sum_{i=1}^{\infty} a_i\, \beta_i$.

Arbitrage. There are no arbitrage opportunities if and only if V is a strictly positive linear functional. The following conditions are necessary and sufficient for the absence of arbitrage:

$$\beta_n > 0 \quad \forall\, n \in \mathbb{N} \qquad \text{and} \qquad \overline{\beta} \leqslant B. \tag{4.3}$$

We address necessity first. If $\beta_n \leqslant 0$, then $z = B\, \boldsymbol{e}_n - \beta_n\, \mathbf{1}_{\mathbb{N}}$ is an arbitrage since $V(z) = 0$ and z is nonnegative and not zero. On the other hand, if $\overline{\beta} > B$, then there is some finite n for which $\sum_{i=1}^{n} \beta_i = b > B$. In this case,

$$z = b\, \mathbf{1}_{\mathbb{N}} - B \sum_{i=1}^{n} \boldsymbol{e}_i = \sum_{i=1}^{n} (b - B)\, \boldsymbol{e}_i + b \sum_{i=n+1}^{\infty} \boldsymbol{e}_i$$

is an arbitrage since $V(z) = 0$ and z is nonnegative and not zero.

We now show that conditions (4.3) are sufficient to guarantee the absence of arbitrage by displaying the requisite linear functional. For $z \in C(\mathbb{N}_\infty) \supset \mathbb{M}$, we can

[28]In Werner (1997), $\beta_i = (2\, i\, (i + 1))^{-1}$, $\overline{\beta} = 1/2$, and $B = 1$.

express V as the strictly positive linear functional[29]

$$V(z) = \int_X z\,d\lambda + \operatorname{Lim}[z]\,(B - \overline{\beta}), \tag{4.4}$$

where $\lambda = \sum_{i \in \mathbb{N}} \beta_i\,\delta_i$. Consequently, there are no arbitrage opportunities in the marketed subspace $\mathbb{M}$. This is true even though $\sum_{i=1}^{\infty} V(\boldsymbol{e}_i) < V\left(\sum_{i=1}^{\infty} \boldsymbol{e}_i\right)$.

Approximate arbitrage. There are no approximate arbitrage opportunities if and only if V is a strictly positive continuous linear functional on $\overline{\mathbb{M}}$, the closure of the space of marketed payouts. For simplicity of exposition, we adopt the norm closure: $\overline{\mathbb{M}} = L_1(\lambda)$. (Essentially the same arguments hold for the weak* closure.)

The following conditions are necessary and sufficient for the absence of approximate arbitrage:

$$\beta_n > 0 \quad \forall\, n \in \mathbb{N} \qquad \text{and} \qquad \overline{\beta} = B. \tag{4.5}$$

We address sufficiency first. If $\overline{\beta} = B$, then $V(z) = \int_X z\,d\lambda$ is a valuation operator on $(L_1(\lambda), \|\,\|_1^{\lambda})$ and hence guarantees the absence of approximate arbitrage opportunities.

We show $\overline{\beta} = B$ is necessary by presenting an approximate arbitrage if (4.3) holds and $\overline{\beta} < B$. Let $z_n = (1 + \gamma_n) \sum_{i=1}^{n} \boldsymbol{e}_i - \gamma_n\,\mathbf{1}_{\mathbb{N}}$, where

$$\gamma_n := \frac{\sum_{i=1}^{n} \beta_i}{B - \sum_{i=1}^{n} \beta_i}.$$

Then $V(z_n) = 0$ and $z_n \xrightarrow{\lambda} \mathbf{1}_{\mathbb{N}}$. There is an approximate arbitrage if $\|z_n - \mathbf{1}_{\mathbb{N}}\|_1^{\lambda} \to 0$. Note

$$\|z_n - \mathbf{1}_{\mathbb{N}}\|_1^{\lambda} = (1 + \gamma_n)\,\widetilde{\beta}_n = \left(\frac{\overline{\beta} - \sum_{i=1}^{n} \beta_i}{B - \sum_{i=1}^{n} \beta_i}\right) B. \tag{4.6}$$

Therefore,

$$\lim_{n \to \infty} \|z_n - \mathbf{1}_{\mathbb{N}}\|_1^{\lambda} = \begin{cases} 0 & \overline{\beta} < B \\ B & \overline{\beta} = B. \end{cases} \tag{4.7}$$

Consequently, $\overline{\beta} < B \implies \|z_n - \mathbf{1}_{\mathbb{N}}\|_1^{\lambda} \to 0$ and $\{z_n\}$ constitutes an approximate arbitrage.[30]

5. Dynamic securities market model in the countable setting

We build on the setting in Section 4, adding a filtration. This is essentially the stochastic setting in the example of Back and Pliska (1991).

We formalize the setting for the doubling strategy as outlined by Harrison and Kreps (1979). We consider a roulette wheel with two colors, RED and BLACK, that will be spun repeatedly. There is positive probability that each color will occur on the next spin.

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[29] Note $a_0\,B + \sum_{i=1}^{\infty} a_i\,\beta_i = \sum_{i=1}^{\infty} (a_0 + a_i)\,\beta_i + a_0\,(B - \overline{\beta})$ where $a_0 = \operatorname{Lim}[z]$ for $z \in \mathbb{M}$.

[30] If the bond included a payout at infinity (in addition to its payouts at finite times), then $\overline{\beta} < B$ would not necessarily generate an approximate arbitrage. See Appendix B.

Stochastic processes. Given the measurable space $(\mathbb{N}_\infty, \mathcal{B}(\mathbb{N}_\infty))$, a random variable Z is a measurable function $Z : \mathbb{N}_\infty \to \mathbb{R}$. Such a random variable can be characterized by the sequence $\{Z(\omega)\}_{\omega\in\mathbb{N}_\infty}$, where $Z(\omega)$ is the value of Z given that RED occurs on the ω-th spin for $\omega \in \mathbb{N}$ and $Z(\infty)$ is the value of Z if RED never occurs. Given S, a collection of subsets of $\mathbb{N}_\infty$, let $\sigma(S)$ denote the σ-algebra generated by S. Define

$$\mathcal{F}_i := \sigma\big(\{\{1\},\{2\},\cdots,\{i\}\}\big) \qquad i \in \mathbb{N}. \tag{5.1}$$

Note $\{\mathcal{F}_i\}_{i=1}^\infty$ is a filtration—an increasing family of sub-σ-algebras such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all $n \in \mathbb{N}$. Let $\mathcal{F}_0 := \sigma(\{\varnothing\}) = \{\varnothing, \mathbb{N}_\infty\}$ and $\mathcal{F}_\infty := \sigma\left(\cup_{i=1}^\infty \mathcal{F}_i\right) = \mathcal{B}(\mathbb{N}_\infty)$.

A stochastic process $\mathcal{Z} = \{Z_i\}_{i=1}^\infty$ is an infinite sequence of random variables. For fixed i, Z_i describes the value of the process $\mathcal{Z}$ after the i-th spin. We restrict our attention to stochastic processes that are adapted to the filtration—that is, such that for each $i \geq 1$, Z_i is measurable with respect to $\mathcal{F}_i$. Note that $Z_i \in C(\mathbb{N}_\infty)$, since Z_i has a constant tail. Every adapted stochastic process $\mathcal{Z}$ can be described in terms of two sequences: $\{z_r(\omega)\}_{\omega\in\mathbb{N}}$ and $\{z_b(i)\}_{i\in\mathbb{N}}$, where $z_r(\omega)$ is the terminal value of $\mathcal{Z}$ given the first RED occurs on spin ω and $z_b(i)$ is the value of $\mathcal{Z}$ after the i-th spin given that RED has not yet occurred:

$$Z_i(\omega) = \begin{cases} z_r(\omega) & \omega \leqslant i \\ z_b(i) & \omega > i. \end{cases} \tag{5.2}$$

Note that if $z_b(i) = z_r(i)$, then Z_i is $\mathcal{F}_{i-1}$-measurable. If $z_b(i) = z_r(i) = \zeta_i$ for all $i \in \mathbb{N}$, then we say that $\mathcal{Z}$ is *predictable* (or *previsible*) and we say the sequence $\{\zeta_i\}_{i\in\mathbb{N}}$ *represents* a predicable stochastic process.

For fixed i, $Z_i(\cdot)$ is a random variable; for fixed ω, $Z.(\omega)$ is a *path*. On each path, $Z_i(\omega)$ converges to $z_r(\omega)$. We can also express Z_i in terms of unit coordinate vectors:

$$Z_i = \sum_{j=1}^{i} z_r(j)\,\boldsymbol{e}_j + z_b(i)\,\widetilde{\boldsymbol{e}}_i. \tag{5.3}$$

Given (5.3), we have

$$\Delta Z_i = \big(z_r(i) - z_b(i-1)\big)\,\boldsymbol{e}_i + \big(z_b(i) - z_b(i-1)\big)\,\widetilde{\boldsymbol{e}}_n, \tag{5.4}$$

where $\Delta Z_i := Z_i - Z_{i-1}$.

Fix the probability measure $\varphi := \sum_{i=1}^\infty \alpha_i\,\delta_i$, where $\alpha_i > 0$ for all $n \in \mathbb{N}$ and $\sum_{i=1}^\infty \alpha_i = 1$. Let $\widetilde{\alpha}_n := \sum_{i=n+1}^\infty \alpha_i$. Note φ is equivalent to λ ($\varphi \ll \lambda$ and $\lambda \ll \varphi$). In particular, $B_\varphi = B_\lambda$, the φ-null sets are $\varnothing$ and $\{\infty\}$, and $\operatorname{supp}\varphi = \mathbb{N}_\infty$. Note $(\mathbb{N}_\infty, \mathcal{B}(\mathbb{N}_\infty), \varphi)$ is a probability space and $(\mathbb{N}_\infty, \mathcal{B}(\mathbb{N}_\infty), \{\mathcal{F}_i\}_{i=1}^\infty, \varphi)$ is a filtered probability space. Define $L_1(\varphi) := L_1(\mathbb{N}_\infty, \mathcal{B}(\mathbb{N}_\infty), \varphi)$. For $\mu \in B_\varphi$, $T_\varphi(\mu) = d\mu/d\varphi = \sum_{i=1}^\infty (m_i/\alpha_i)\,\boldsymbol{e}_i$. For $z \in L_1(\varphi)$, $T_\varphi^{-1}(z) = \sum_{i=1}^\infty \alpha_i\,z(i)\,\delta_i$. Let $\pi = T_\varphi(\lambda) = d\lambda/d\varphi = \sum_{i=1}^\infty (\beta_i/\alpha_i)\,\boldsymbol{e}_i$. Note $E^\varphi[\pi] = \sum_{i=1}^\infty \pi(i)\,\alpha_i = \overline{\beta}$. Let $E^\varphi[Z] = \sum_{i=1}^\infty Z(i)\,\alpha_i$ for $Z \in L_1(\varphi)$. Let $E^\varphi_{i-1}[Z] := E^\varphi[Z \mid \mathcal{F}_{i-1}]$ for $\mathcal{F}_{i-1}$-measurable $Z \in L_1(\varphi)$. Finally, let p_i denote the conditional probability of $\omega = i$

given $\omega \geq i$:

$$p_i = \frac{\alpha_i}{1 - \sum_{j=1}^{i-1} \alpha_j} = \frac{\alpha_i}{\widetilde{\alpha}_i} \qquad \text{and} \qquad \alpha_i = p_i \prod_{j=1}^{i-1} (1 - p_j).$$

(α_i is the unconditional probability of $\omega = i$.)

Now we construct a *shock process* $\mathcal{U} := \{U_i\}_{i=0}^{\infty}$ to use as a building block. Define $\zeta_i := \sqrt{(1 - p_i)/p_i}$. Let $U_i := \sum_{j=1}^{i} u_r(j)\, \boldsymbol{e}_j + u_b(i)\, \widetilde{\boldsymbol{e}}_i$, where

$$\begin{aligned} u_r(i) &= u_b(i-1) + \zeta_i \\ u_b(i) &= u_b(i-1) - \zeta_i^{-1} \end{aligned}$$

subject to $u_b(0) = 0$. Note $\Delta U_i = \zeta_i\, \boldsymbol{e}_i - \zeta_i^{-1}\, \widetilde{\boldsymbol{e}}_i$. $\mathcal{U}$ is a φ-martingale: $E_{i-1}^{\varphi}[\Delta U_i] = 0$. In addition,

$$E_{i-1}^{\varphi}[(\Delta U_i)^2](\omega) = \begin{cases} 0 & \omega < i \\ 1 & \omega \geq i. \end{cases}$$

Constructing stochastic processes from measures. Given a measure $\mu \in M(X)$, we construct a stochastic process adapted to the filtration $\{\mu_n\}_{n=1}^{\infty}$ such that $\mu_n \xrightarrow{w^*} \mu$.

Note that the only nonempty λ-null set is $\{\infty\}$ and that $\{\infty\} \notin \mathcal{F}_n$ for any $n \in \mathbb{N}$.[31] Therefore, given any $\mu \in M(X)$, the restriction of μ to $\mathcal{F}_n$ is absolutely continuous with respect to the restriction of λ to $\mathcal{F}_n$.

Let

$$\mu := \sum_{i=1}^{\infty} m_i\, \delta_i + m_\infty\, \delta_\infty.$$

The restrictions of μ and λ to $\mathcal{F}_n$ are given by

$$\lambda_n = \sum_{i=1}^{n} \beta_i\, \delta_i + \widetilde{\beta}_n\, \widetilde{\delta}_n$$

and

$$\hat{\mu}_n = \sum_{i=1}^{n} m_i\, \delta_i + (\widetilde{m}_n + m_\infty)\, \widetilde{\delta}_n, \tag{5.5}$$

where $\widetilde{m}_n := \sum_{i=n+1}^{\infty} m_i$. Then

$$z_n = \frac{d\hat{\mu}_n}{d\lambda_n} = \frac{d\hat{\mu}_n}{d\lambda_n} = \sum_{i=1}^{n} \left(\frac{m_i}{\beta_i} \right) \boldsymbol{e}_i + \left(\frac{\widetilde{m}_n + m_\infty}{\widetilde{\beta}_n} \right) \widetilde{\boldsymbol{e}}_n \tag{5.6}$$

and[32]

$$\mu_n = T_\lambda^{-1}(z_n) = \hat{\mu}_n.$$

[31] Similar comments apply to any positive measure equivalent to λ such as φ.

[32] Note that μ_n is the extension of $\hat{\mu}_n$ to $\mathcal{B}(\mathbb{N}_\infty)$, where $\hat{\mu}_n$ is the restriction of μ to $\mathcal{F}_n$. In this setting, it is not necessary to compute $\hat{\mu}_n$ first.

In addition, define

$$z := \sum_{i=1}^{\infty} \left(\frac{m_i}{\beta_i}\right) \boldsymbol{e}_i,$$

where $z \in L_1(\lambda)$. Note $\{z_n\}_{n=1}^{\infty}$ is adapted to the filtration, $\|z_n\|_1^{\lambda} = \|\mu\|$ for all $n \in \mathbb{N}$, and

$$z_{n-1}(\omega) = \begin{cases} z_n(\omega) & \omega \leqslant n-1 \\ \left(\frac{\beta_n}{\widetilde{\beta}_{n-1}}\right) z_n(n-1) + \left(\frac{\widetilde{\beta}_n}{\widetilde{\beta}_{n-1}}\right) z_n(n) & \omega > n-1. \end{cases} \tag{5.7}$$

Thus, if $\overline{\beta} = 1$, then λ is a probability measure and $\{z_n\}_{n=1}^{\infty}$ is a martingale where $E^{\lambda}[z_n] = \mu(\mathbb{N}_{\infty})$ and

$$z_n = E^{\lambda}[z|\mathcal{F}_n] + \left(\frac{m_{\infty}}{\widetilde{\beta}_n}\right) \widetilde{\boldsymbol{e}}_n.$$

Note $z_n \xrightarrow{\lambda\text{-a.e.}} z$ and $\|z - z_n\|_1^{\lambda} = \|\mu - \mu_n\| \underset{n}{\rightarrow} 2\,|m_{\infty}|$. Note $\{z_n\}_{n=1}^{\infty}$ is uniformly integrable if and only if $m_{\infty} \neq 0$, since $\lim_{n\to\infty}\ |m_{\infty}|/\widetilde{\beta}_n = \infty$ and $E^{\lambda}[|m_{\infty}|/\widetilde{\beta}_n] = |m_{\infty}|$.

Given $\pi = d\lambda/d\varphi = \sum_{i=1}^{\infty}(\beta_i/\alpha_i)\,\boldsymbol{e}_i$, define

$$\pi_n := E^{\varphi}[\pi|\mathcal{F}_n] = \sum_{i=1}^{n} \left(\frac{\beta_i}{\alpha_i}\right) \boldsymbol{e}_i + \left(\frac{\widetilde{\beta}_n}{\widetilde{\alpha}_n}\right) \widetilde{\boldsymbol{e}}_n.$$

Note $\{\pi_n\}_{n=1}^{\infty}$ is a φ-martingale and $E^{\varphi}[\pi_n] = \overline{\beta}$ for all $n \in \mathbb{N}$. Also note $\{z_n\,\pi_n\}_{n=1}^{\infty}$ is a φ-martingale and

$$z_n\,\pi_n = E[z\,\pi|\mathcal{F}_n] + \left(\frac{m_{\infty}}{\widetilde{\alpha}_n}\right) \widetilde{\boldsymbol{e}}_n = \sum_{i=1}^{n} \left(\frac{m_i}{\alpha_i}\right) \boldsymbol{e}_{\boldsymbol{i}} + \left(\frac{\widetilde{m}_n + m_{\infty}}{\widetilde{\alpha}_n}\right) \widetilde{\boldsymbol{e}}_n,$$

where $z\,\pi = \sum_{i=1}^{\infty}(m_i/\alpha_i)\,\boldsymbol{e}_i$.

Given $\xi = \lambda + (1-\overline{\beta})\,\delta_{\infty}$, let

$$\xi_n = \sum_{i=1}^{n} \beta_i\,\delta_i + \big(\widetilde{\beta}_n + (1-\overline{\beta})\big)\,\widetilde{\delta}_n.$$

Note (for all $n \in \mathbb{N} \cup \{0\}$) $\xi_n(A) = \xi(A)$ for all $A \in \mathcal{B}(X)$. In particular, $\xi_n(X) = 1$. Now let $\mathcal{Y} = \{Y_n\}_{n=0}^{\infty}$ where

$$Y_n = T_{\varphi}(\xi_n) = \sum_{i=1}^{n} \left(\frac{\beta_i}{\alpha_i}\right) \boldsymbol{e}_i + \left(\frac{\widetilde{\beta}_n + (1-\overline{\beta})}{\widetilde{a}_n}\right) \widetilde{\boldsymbol{e}}_n.$$

In particular, $Y_0 = \widetilde{\boldsymbol{e}}_0$. By construction, $\mathcal{Y} \subset L_1(\varphi)$ is a strictly positive martingale and $Y_n \xrightarrow{w^*} \xi$, but $\mathcal{Y}$ is UI if and only if $\overline{\beta} = 1$ (i.e., $\xi = \lambda$).

Moreover,

$$\int_X d\mu = \int_X \left(\frac{d\mu}{d\xi}\right) d\xi = \int_X \left(\frac{d\mu_n}{d\xi_n}\right) d\xi_n = E^{\xi}\left[\frac{d\mu_n}{d\xi_n}|\mathcal{F}_n\right]$$

Dynamic securities market. Consider a market for trading two securities at a countable number of times $0 = t_0 < t_1 < \cdots < T$.[33] The price of one security (the bond or 'money-market account') at time t_i equals 1 in every state: $B_i(\omega) = 1$. The state-by-state price of the second security (the 'stock') at time t_i is denoted $S_i(\omega)$.[34]

The dynamics of the stock price can be specified by

$$\Delta S_i = m_i + \sigma_i \, \Delta U_i, \tag{5.8}$$

given some S_0. We assume $\sigma_i \neq 0$ for all $i \in \mathbb{N}$ (spanning condition). The mean and the variance of the conditional change in the stock price (conditional on $\omega \geq i$) are given by

$$E^{\varphi}_{i-1}[\Delta S_i](\omega \geq i) = m_i \qquad \text{and} \qquad E^{\varphi}_{i-1}[(\Delta S_i - m_i)^2](\omega \geq i) = \sigma_i^2.$$

Given (5.8), we have

$$S_i = S_0 + \sum_{j=1}^{i} \Delta S_i = S_0 + \sum_{j=1}^{i} m_j + \sum_{j=1}^{i} \sigma_j \, \Delta U_j$$

so that

$$\begin{aligned} s_r(i) &= s_b(0) + \sum_{j=1}^{i} m_j - \sum_{j=1}^{i-1} \sigma_j \, \zeta_j^{-1} + \sigma_i \, \zeta_i \\ s_b(i) &= s_b(0) + \sum_{j=1}^{i} m_j - \sum_{j=1}^{i} \sigma_j \, \zeta_j^{-1}. \end{aligned}$$

Price of risk, state-price deflator, and change-of-measure process. Let $\ell_i := m_i/\sigma_i$, which is the coefficient of variation (the ratio of the mean to the standard deviation) conditional on $\omega \geq i$. Since the interest rate is zero, ℓ_i is the *Sharpe ratio* of the stock. It is also the *price of risk*. (Note $\{m_i\}_{i=1}^{\infty}$, $\{\sigma_i\}_{i=1}^{\infty}$, and $\{\ell_i\}_{i=1}^{\infty}$ represent predictable processes.)

Define $\mathcal{Y} := \{Y_i\}_{i=1}^{\infty}$. Let $Y_i := \prod_{j=1}^{i} X_i$ where $X_i := 1 - \ell_i \, \Delta U_i$. Note $E^{\varphi}_{i-1}[X_i] = 1$ and $E^{\varphi}_{i-1}[X_i \, \Delta U_i](\omega \geq i) = -\ell_i$ since $X_i \, \Delta U_i = \Delta U_i - \ell_i \, (\Delta U_i)^2$. Note that $\mathcal{Y}$ is a φ-martingale: $E^{\varphi}_{i-1}[\Delta Y_i] = 0$ since $\Delta Y_i = -Y_{i-1} \, \ell_i \, \Delta U_i$. In particular, $E^{\varphi}[Y_n] = 1$ for all $n \in \mathbb{N}$.

Note that $\{S_i \, Y_i\}_{i=1}^{\infty}$ is also a φ-martingale:

$$E^{\varphi}_{i-1}[\Delta(S_i \, Y_i)] = Y_{i-1} \, E^{\varphi}_{i-1}[X_i \, \Delta S_i] + S_{i-1} \, E^{\varphi}_{i-1}[\Delta Y_i] = 0,$$

since (conditional on $\omega \geq i$)

$$E^{\varphi}_{i-1}[X_i \, \Delta S_i] = m_i + \sigma_i \, E^{\varphi}_{i-1}[X_i \, \Delta U_i] = m_i - \ell_i \, \sigma_i = 0.$$

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[33]For example, $T < \infty$ and $t_i = T\,(1 - (1/2)^i)$.

[34]To allow for a non-zero interest rate r, let $\widetilde{B}_i(\omega) = e^{r\,t_i}$ be the money-market account and define $\widetilde{S}_i := \widetilde{B}_i \, S_i$. Then $B_i = \widetilde{B}_i/\widetilde{B}_i$ and $S_i = \widetilde{S}_i/\widetilde{B}_i$.

It is convenient to reexpress X_i and Y_n. In particular,

$$X_i = \sum_{\omega=1}^{i-1} \boldsymbol{e}_\omega + \left(\frac{q_i}{p_i}\right) \boldsymbol{e}_i + \left(\frac{1-q_i}{1-p_i}\right) \widetilde{\boldsymbol{e}}_i, \tag{5.9}$$

where

$$q_i := (1 - \ell_i\, \zeta_i)\, p_i. \tag{5.10}$$

Thus $\ell_i = (p_i - q_i)/\sqrt{p_i\,(1-p_i)}$. Note that q_i can be computed directly from s_r and s_b:

$$q_i = \frac{s_b(i) - s_b(i-1)}{s_b(i) - s_r(i)}.$$

The condition $\sigma_i \neq 0$ is equivalent to $s_b(i) - s_r(i) \neq 0$.[35] Given (5.9), we can write

$$Y_n = \sum_{i=1}^{n} y_r(i)\, \boldsymbol{e}_i + y_b(n)\, \widetilde{\boldsymbol{e}}_n,$$

where

$$\begin{aligned} y_r(i) &= \frac{q_i \prod_{j=1}^{i-1}(1-q_j)}{p_i \prod_{j=1}^{i-1}(1-p_j)} = \frac{\beta_i}{\alpha_i} \\ y_b(n) &= \frac{\prod_{i=1}^{n}(1-q_i)}{\prod_{i=1}^{n}(1-p_i)} = \frac{\widetilde{\beta}_n + (1-\overline{\beta})}{\widetilde{\alpha}_n}, \end{aligned}$$

using $\beta_i = q_i \prod_{j=1}^{i-1}(1-q_j)$.

If $\mathcal{Y}$ is strictly positive martingale, then it is a state-price deflator.[36] It is easily seen that $\mathcal{Y}$ is strictly positive if and only if

$$0 < q_i < 1 \qquad \forall\, i \in \mathbb{N}. \tag{5.11}$$

Note (5.11) is equivalent to (4.3). If Condition (5.11) holds, we can compute the equivalent measure $\lambda = T_\varphi^{-1}(y_r)$, where $y_r = \pi$. (Note $Y_n \xrightarrow{\varphi\text{-a.e.}} \pi$.)

A uniformly integrable state-price deflator is a change-of-measure process. Note

$$T_\varphi^{-1}(Y_n) = \sum_{i=1}^{n} \beta_i\, \delta_i + \big(\widetilde{\beta}_n + (1-\overline{\beta})\big) \sum_{i=n+1}^{\infty} \left(\frac{\alpha_i}{\widetilde{\alpha}_n}\right) \delta_i,$$

and therefore

$$T_\varphi^{-1}(Y_n) \xrightarrow{w^*} \xi, \tag{5.12}$$

[35]Given s_r and s_b, we can write

$$\begin{aligned} m_i &= \big(s_r(i) - s_b(i-1)\big)\, p_i + \big(s_b(i) - s_b(i-1)\big)\,(1-p_i) \\ &= (p_i - q_i)\,\big(s_b(i) - s_r(i)\big) \\ \sigma_i &= \big\{\big(s_r(i) - s_b(i-1) - m_i\big)^2\, p_i + \big(s_b(i) - s_b(i-1) - m_i\big)^2\,(1-p_i)\big\}^{1/2} \\ &= \big\{p_i\,(1-p_i)\,\big(s_b(i) - s_r(i)\big)^2\big\}^{1/2}. \end{aligned}$$

[36]Warning: Harrison and Kreps (1979) and others include uniform integrability in their definition of a state-price deflator (which we call a change-of-measure process).

where $\xi = \lambda + (1 - \overline{\beta})\,\delta_\infty$ [see (B.1)]. Therefore, $\mathcal{Y}$ is UI if and only if $\overline{\beta} = 1$. Given $\overline{\beta} = 1$, $E^\lambda[Z] = E^\varphi[y_r\, Z]$ and $E^\lambda_{i-1}[Z_i] = E^\lambda_{i-1}[X_i\, Z_i]$. In this case, we say λ is an equivalent martingale measure.

Self-financing trading strategies. A trading strategy is a pair of predictable processes represented by $(\{\theta_i\}_{i=1}^\infty, \{\phi_i\}_{i=1}^\infty)$ where θ_i is the number of bonds and ϕ_i is the number of shares of stock held at time t_i after any changes in the value of the stock but before any rebalancing. The exposition that follows is intended to make the meaning of this clear.

Let G_0 denote the initial amount invested. At time t_0, G_0 is apportioned between the stock and the bond: $G_0 = \theta_1 + \phi_1\, S_0$. At time t_1, after any changes in the share price but before any rebalancing, the value of the portfolio is $G_1 = \theta_1 + \phi_1\, S_1$. After rebalancing, the value of the portfolio is $G_1' = \theta_2 + \phi_2\, S_1$. In gereral, $G_i = \theta_i + \phi_i\, S_i$ and $G_i' = \theta_{i+1} + \phi_{i+1}\, S_i$. For a self-financing trading strategy, $G_i' = G_i$ which implies

$$(\theta_{i+1} - \theta_i) + (\phi_{i+1} - \phi_i)\, S_i = 0 \qquad \forall\, i \in \mathbb{N}. \tag{5.13}$$

In other words, any change in the value of the stock holdings that comes from rebalancing is offset by an equal change in the opposite direction of the value of the bond holdings. From one period to the next, the change in the value of a portfolio generated by a self-financing trading strategy is $\Delta G_i = \phi_i\, \Delta S_i$ and therefore $G_i = G_0 + \sum_{j=1}^i \phi_j\, \Delta S_j$. We refer to G_i as the *gain* and $\mathcal{G} = \{G_i\}_{i=1}^\infty$ as the gains process.[37] A self-financing trading strategy is characterized by the pair $(G_0, \{\phi_i\}_{i=1}^\infty)$.

A *finite trading strategy* is a self-financing trading strategy for which $\phi_i = 0$ for all $i > n \in \mathbb{N}$. The gain G_n generated by a finite trading strategy is called a *finite gain* stopped at n. Let $G_i = \sum_{j=1}^i g_r(j)\, \boldsymbol{e}_j + g_b(i)\, \widetilde{\boldsymbol{e}}_i$, where

$$g_r(i) = g_b(i-1) + \phi_i\,\big(s_r(i) - s_b(i-1)\big) \tag{5.14a}$$

$$g_b(i) = g_b(i-1) + \phi_i\,\big(s_b(i) - s_b(i-1)\big), \tag{5.14b}$$

subject to $g_b(0) = G_0$. For a finite trading strategy stopped at n, $G_i = g_r$ for all $i \geq n$. Consequently, $G_i \to g_r$ in all modes of convergence.

We now describe the space of marketed payouts and show that it is the space of finite gains. Define $\mathbb{M}_{0,n} := B \cup (\bigcup_{0<i\leqslant n} S_i)$ and the set of marketed securities as $\mathbb{M}_0 := \bigcup_{n\geq 0} \mathbb{M}_{0,n}$. The given prices for the marketed securities are $V_0(B) = 1$ and $V_0(S_i) = S_0$ for all $i \in \mathbb{N}$. The space of marketed payouts, $\mathbb{M} := \mathbf{sp}(\mathbb{M}_0)$. Now $z \in \mathbf{sp}(\mathbb{M}_{0,n})$ has the form $z = a_0\, B + \sum_{i=1}^n a_i\, S_i$, where $V(z) = a_0\, V_0(B) + \sum_{i=1}^n a_i\, V_0(S_i) = a_0 + S_0 \sum_{i=1}^n a_i$. This z can be obtained via the following finite self-financing trading strategy: $G_0 = V(z)$ and $\phi_i = \sum_{j=i}^n a_j$.

It turns out that set of marketed securities in the dynamic case spans the same space as the set of marketed securities in the static case above, so that the two spaces of marketed payouts are identical. Therefore, it is not surprising that the conditions for no arbitrage and no approximate arbitrage are also identical.

[37] Warning: Pliska (1997) and others refer to $G_i - G_0$ as the gain.

Arbitrage. An arbitrage is a finite trading strategy for which $G_0 = 0$, $g_r(i) \geq 0$ for all $i \in \mathbb{N}$, and and $g_r(i) > 0$ for some $i \in \mathbb{N}$.

Consider the finite trading strategy where $G_0 = 0$ and

$$\phi_i = \begin{cases} \frac{1}{s_r(n) - s_b(n)} & i = n \\ 0 & i \neq n. \end{cases}$$

This trading strategy produces $g_r = (1 - q_n)\, \boldsymbol{e}_n - q_n\, \widetilde{\boldsymbol{e}}_n$, which is an arbitrage unless Condition (5.11) holds.

Conversely, suppose Condition (5.11) holds. Set $\beta_i = q_i \prod_{j=1}^{i-1}(1 - q_j)$.[38] Then for any finite gain stopped at n, we can use (5.14) to eliminate $\{\phi_i\}_{i=1}^{n}$ and $\{g_b(i)\}_{i=1}^{n-1}$ to produce:

$$\sum_{i=1}^{n} g_r(i)\, \beta_i + g_b(n)\, \big(\widetilde{\beta}_n + (1 - \overline{\beta})\big) = G_0, \tag{5.15}$$

where $g_r(j) = g_b(n)$ for $j \geq n+1$. Given the positivity of β_i and $\widetilde{\beta}_n + (1 - \overline{\beta})$, no finite gain can be an arbitrage. Note that we can express (5.15) as $V(g_r) = G_0$, where V is given in (4.4).

Approximate arbitrage. Assume the no-arbitrage Condition (5.11) holds. Therefore, a state-price deflator exists and $\lambda = \sum_{i=1}^{\infty} \beta_i\, \delta_i$ is an equivalent measure. (See above.) Given $\overline{\beta} = 1$, λ is an equivalent martingale measure and the gains process is a martingale: $E_{i-1}^{\lambda}[\Delta G_i] = \phi_i\, E_{i-1}^{\lambda}[\Delta S_i] = \phi_i\, E_{i-1}^{\varphi}[X_i\, \Delta S_i] = 0$. Therefore, $E^{\lambda}[G_n] = G_0$ where $E^{\lambda}[G_n] = \int_X G_n\, d\lambda$.

An *admissible* self-financing trading strategy is one for which the generated gain process converges in the appropriate topology. For example, if the payout space is $(L_1(\lambda), \|\,\|_1^{\lambda})$, then an admissible trading strategy is one for which $\|G_n - G\|_1^{\lambda} \to 0$ for some $G \in L_1(\lambda)$. In this case, $z \mapsto E^{\lambda}[z]$ is a valuation operator. On the other hand, if the payout space is $(M(X), w^*)$, then an admissible trading strategy is one for which $T_{\lambda}^{-1}(G_n) \xrightarrow{w^*} \mu \in M(X)$, where $T_{\lambda}(\mu) = d\mu/d\lambda$. In this case $\mu \mapsto \langle \mathbf{1}, \mu \rangle$ is a valuation operator. For $\mu \in B_{\lambda}$, $\langle \mathbf{1}, \mu \rangle = E^{\lambda}[T_{\lambda}^{-1}(\mu)]$.

Equivalences. The following statements are equivalent:

(1) $\mathcal{Y}$ is a strictly positive and uniformly integrable φ-martingale.
(2) There is an equivalent martingale measure λ.
(3) $z \mapsto \int_X z\, d\lambda$ is a valuation operator for $(L_1(\lambda), \|\,\|_1^{\lambda})$.

6. Role reversal in the countable setting

In this section, we reverse the roles of the dual pair of spaces: Let $C(\mathbb{N}_\infty)$ equipped with the sup norm topology be the space of payouts and let $M(\mathbb{N}_\infty)$ be the space

[38] Note q_n is the *conditional price* of bet that pays \$1 if RED occurs (conditional on $\omega \geq n$), while β_n is the *unconditional price* (i.e., the price given before the first spin; conditional on $\omega \geq 1$).

of price systems. The valuation operator is given by

$$f \mapsto \langle f, \mu \rangle = \sum_{i=1}^{\infty} f(i)\, m_i + \mathrm{Lim}[f]\, m_\infty,$$

for some strictly positive μ where $m_i > 0$ for $i \in \mathbb{N}_\infty$.

Static setting. Let us revisit the conditions for no arbitrage and no approximate arbitrage in the static setting of Section 4. In this case, Conditions (4.3) are necessary and sufficient for the existence of a valuation operator.

Take $\xi = \lambda + (1 - \overline{\beta})\, \delta_\infty$ [as given in (B.1)] to represent the *price system*. Then, given Conditions (4.3), $V : C(\mathbb{N}_\infty) \to \mathbb{R}$ is a valuation operator, where

$$V(f) = \langle f, \xi \rangle = \langle f, \lambda \rangle + (1 - \overline{\beta})\, \langle f, \delta_\infty \rangle = \int_X f\, d\lambda + \mathrm{Lim}[f]\, (1 - \overline{\beta}).$$

Compare with (4.4). The payout to the n-th Arrow–Debreu security is $\boldsymbol{e}_n$ and the payout to the bond be given by $\mathbf{1}$. We have $\langle \boldsymbol{e}_n, \xi \rangle = \beta_n$ and $\langle \mathbf{1}, \xi \rangle = 1$.

Dynamic setting. In this setting, a trading strategy is admissible if and only if it generates a gain process that converges in the sup norm topology.[39]

Note $G_n \in C(\mathbb{N}_\infty)$, assuming $G_n(\infty) = \lim_{i\to\infty} G_n(i)$. If $\{G_n\}_{n=1}^{\infty}$ converges in the sup norm topology, then $G_n \xrightarrow{\|\,\|_\infty} g_r \in C(\mathbb{N}_\infty)$, in which case $\lim_{n\to\infty} g_r(n) - g_b(n) = 0$. Given G_0 and g_r, we can eliminate ϕ_i from (5.14) and solve recursively for

$$g_b(i) = \frac{G_0 - \sum_{j=1}^{i} g_r(j)\, \beta_j}{\widetilde{\beta}_i + (1 - \overline{\beta})}. \tag{6.1}$$

Assuming (5.11) and referring to (6.1), note

$$\lim_{n\to\infty} g_r(n) - g_b(n) = \mathrm{Lim}[g_r] - \frac{G_0 - \sum_{i=1}^{\infty} g_r(i)\, \beta_i}{1 - \overline{\beta}}.$$

Therefore

$$G_n \xrightarrow{\|\,\|_\infty} g_r \iff \langle g_r, \xi \rangle = G_0.$$

Consequently, $g_r = \mathbf{1}$ combined with $G_0 = 0$ does not produce an approximate arbitrage because the implied gains process does not converge. Thus $f \mapsto \langle f, \xi \rangle$ is a valuation operator.

In this case, $\mathcal{Y}$ is a state-price deflator, although it is not uniformly integrable as can be seen in (5.12) and it does not deliver a change-of-measure process (as we have defined it). Thus, ξ is a martingale measure, but it is not equivalent to φ. Nevertheless, (i) $\int_X G_n\, d\xi = E^\varphi[G_n\, Y_n]$ and (ii) if $\|G_n - g_r\|_\infty \to 0$, then $\int_X g_r\, d\xi = \lim_{n\to\infty} E^\varphi[G_n\, Y_n]$.

[39] Back and Pliska (1991) present the following stock-price dynamics: $s_r(i) = 2^{-i}\,(i^2 + 2\,i + 2)$ and $s_b(i) = 2^{-i}$. These dynamics produce $q_i = 1/(i+1)^2$, $\beta_i = (2\,i\,(i+1))^{-1}$, and $\overline{\beta} = 1/2$. Other stock-price dynamics produce identical bet prices; for example, $s_r(i) = 2$ and $s_b(i) = 2/(i+2)$. Even though the bet prices are identical, the trading strategies $\{\phi_i\}_{i=1}^{\infty}$ required to produce a given payout are quite different.

Equivalences. The following statements are equivalent:

(1) $\mathcal{Y}$ is a strictly positive φ-martingale.
(2) There is a valuation operator for $(C(X), \| \|_\infty)$.

Appendix A. The Cantor Space

The Cantor space $\mathcal{C} = \{0,1\}^{\mathbb{N}}$ of countably infinite sequences of zeros and ones is compact in the product topology (where $\{0,1\}$ is equipped with the discrete topology).[40] The topology is metrizable.[41] In fact, the Cantor space is the mother of all compact metrizable spaces: Every compact metrizable space is homeomorphic to a quotient space of the Cantor space.[42] We illustrate this with two examples.

First, consider the following equivalence relation: $x \sim y$ if

$$\sum_{i=1}^{\infty} 2^{-i}\, x(i) = \sum_{i=1}^{\infty} 2^{-i}\, y(i).$$

For example, $\{1,0,0,\ldots\} \sim \{0,1,1,\ldots\}$. In this case, $\mathcal{C}/{\sim}$ is homeomorphic to the closed unit interval on the real line $[0,1]$ via the homeomorphism $p : \mathcal{C}/{\sim} \to [0,1]$ where $p([x]) = \sum_{i=1}^{\infty} 2^{-i}\, x(i)$.

Second, consider the following. For $n \in \mathbb{N}$ and $0 \leqslant i \leqslant 2^n - 1$, define

$$B_n^i := \left\{ x \in \mathcal{C} : \sum_{j=1}^{n} 2^{n-j}\, x(j) = i \right\}. \tag{A.1}$$

Let $B_0^0 := \mathcal{C}$ and $B_\infty^1 := \big\{\{0,0,\ldots\}\big\}$. Note $\{B_n^1\}_{n\in\mathbb{N}_\infty}$ is a partition of $\mathcal{C}$. (If $x \in B_n^1$, then $x(n) = 1$ and $x(i) = 0$ for $i < n$.) Define the equivalence relation by $x \sim y$ if $x, y \in B_n^1$. In this case, $\mathcal{C}/{\sim}$ is homeomorphic to $\mathbb{N}_\infty$ via the homeomorphism $p : \mathcal{C}/{\sim} \to \mathbb{N}_\infty$, where $p([x \in B_n^1]) = n$.

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[40] The Cantor set (as it is usually defined) is the following set of points in the unit interval:

$$C = \left\{ \sum_{i=1}^{\infty} 3^{-i}\, a(i) : a(i) = 0 \text{ or } a(i) = 2 \right\}.$$

The Cantor space $\mathcal{C}$ is homeomorphic to the Cantor set C via the homeomorphism $g : \mathcal{C} \to [0,1]$, where $g(x) = \sum_{i=1}^{\infty} 3^{-i}\, 2\, x(i)$.

[41] A metric that generates the product topology $\tau_{\mathcal{C}}$ is

$$d(x,y) = \sum_{i=1}^{\infty} 3^{-i}\, |x(i) - y(i)|,$$

for $x, y \in \mathcal{C}$. Thus $(\mathcal{C}, d)$ is a compact metric space and $x_n \xrightarrow{\tau_{\mathcal{C}}} x \iff d(x_n, x) \to 0$.

[42] Let X be a topological space and let $\sim$ be an equivalence relation on X. Let $X/{\sim}$ denote the set of all equivalence classes $[x] = \{y \in X : x \sim y\}$. $X/{\sim}$ is called the quotient space of X by the equivalence relation $\sim$. Define a function $q : X \to X/{\sim}$ by $q(x) = [x]$. This map q is called the quotient map. Define a set $A \subset X/{\sim}$ to be open if $q^{-1}(A)$ is open in X. This collection of open sets defines a topology on $X/{\sim}$ called the quotient topology. If X is compact, then so is $X/{\sim}$. A function $f : X/{\sim} \to Y$ is continuous if and only if the composite function $f \circ q : X \to Y$ is continuous.

Appendix B. Redefine the bond's payout

If $\overline{\beta} < B$, we can avoid approximate arbitrage opportunities by redefining the bond's payout. (We are not always free to do this.) As a first step, we can identify the payouts of the bond and the A–D securities with elements of B_λ: $T_\lambda^{-1}(\mathbf{1}_\mathbb{N}) = \lambda$ and $T_\lambda^{-1}(\boldsymbol{e}_n) = \beta_n\,\delta_n$. Next, let the payout to the bond be redefined as[43]

$$\xi := \lambda + (B - \overline{\beta})\,\delta_\infty. \tag{B.1}$$

Then $\mu \mapsto \langle \mathbf{1}, \mu\rangle$ is a valuation operator. Note $\langle \mathbf{1}, \beta_n\,\delta_n\rangle = \beta_n$ and $\langle \mathbf{1}, \xi\rangle = \langle \mathbf{1}, \lambda\rangle + (B - \overline{\beta})\,\langle \mathbf{1}, \delta_\infty\rangle = 1$. Note that we can approximate ξ by $\lambda + (B - \overline{\beta})\,\widetilde{\delta}_n$.

It is possible to redefine the payout to the bond in such a way as to achieve the result in the preceeding paragraph without placing any weight directly on $\{\infty\}$. Let $C_b(X)$ denote the space of bounded continuous functions on a normal Hausdorff space X and let $ba_{\mathrm{n}}(\mathcal{A}_X)$ denote the space of normal charges on the algebra generated by the open sets of X. Then a version of the Riesz Representation Theorem states that the dual of $C_b(X)$ is $ba_{\mathrm{n}}(\mathcal{A}_X)$.[44] Since $\mathbb{N}_\infty$ is a normal Hausdorff space and $C(\mathbb{N}_\infty) = C_b(\mathbb{N}_\infty)$, the theorem applies here. The pure charge that represents the linear functional $f \mapsto (B - \overline{\beta})\ \mathrm{Lim}[f]$, for $f \in C(\mathbb{N}_\infty)$, does not involve placing any weight on $\{\infty\}$. Nevertheless, the effect is the same.

Appendix C. Some useful theorems

These theorems are taken from Aliprantis and Border (1999).

Theorem (2.66)**.** Let (X, τ) be a noncompact locally compact Hausdorff space and let $X_\infty = X \cup \{\infty\}$, where $\infty \notin X$. Then the collection

$$\tau_\infty = \tau \cup \{X_\infty \setminus K : K \subset X \text{ is compact}\}$$

is a topology on X_∞. Moreover, (X_∞, τ_∞) is a compact Hausdorff space and X is an open dense subset of X_∞.

Theorem (3.32)**.** The one-point compactification X_∞ of a noncompact locally compact Huasdorff space X is metrizable if and only if X is second countable.

Theorem (8.48)**.** A compact Hausdorff space X is metrizable if and only if $C(X)$ is a separable Banach lattice.

Theorem (6.34)**.** A normed space is separable if and only if the closed unit ball of its dual space is w*-metrizable.

Theorem (13.15)**.** If X is a compact metrizable space . . . , then the norm dual of $C(X)$ can be identified with the . . . [space] of finite Borel measures on X.

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[43] We can think of this as a coupon bond with lump-sum payment at infinity.

[44] Theorem 13.10 in Aliprantis and Border (1999).

Appendix D. Quotations

The following quotes are taken form Aliprantis and Border (1999):

> Topology is the abstract study of convergence and approximation. [p. 19]
>
> There are ... advantages to working with general topological spaces. For example, one can define topologies to make our favorite functions continuous. These are called *weak* topologies. ... [W]eak topologies are fundamental to understanding the equilibria of economies with an infinite dimensional commodity space. [p. 20]
>
> [An] important topological notion is compactness. Compact sets can be approximated arbitrarily well by finite subsets. (In Euclidean spaces, the compact sets are the closed and bounded sets.) ... [T]he Alaolgu Theorem ... describes a general class of compact sets in infinite dimensional spaces. [p. 20]
>
> In many ways compactness can be viewed as a topological generalization of finiteness. [p. 39]
>
> There are two classes of topologies that by and large include everything of interest. The first and most familiar is the class of topologies that are generated by a metric. The second class is the class of weak topologies. [p. 47]
>
> One way to think of functional analysis is as the branch of mathematics that studies the extent to which the properties possessed by finite dimensional spaces generalize to infinite dimensional spaces. [p. 161]

In the introduction to Chapter 5 ("Topological Vector Spaces," p. 163) they write (with minor adjustments to their notation[45])

> [One] of the consequences of the Hahn–Banach Theorem is that the set of continuous linear functionals on a locally convex space separates points. The collection of continuous linear functionals on X is known as the (topological) dual space, denoted X^*. Now each $x \in X$ defines a linear functional on X^* by $x(x^*) = x^*(x)$. Thus we are led to the study of *dual pairs* $\langle X, X^* \rangle$ of spaces and their associated *weak topologies*. These weak topologies are locally convex. The weak topology on X^* induced by X is called the weak-* topology. The most familiar example of a dual pair is probably the pairing of functions and measures—each defines a linear functional via the integral $\int_X f\, d\mu$, which is linear in f for fixed μ, and linear in μ for fixed f. (The weak topology induced on probability measures by this duality with continuous functions is the topology of convergence in

[45]They use X' to denote the topological dual and X^* to denote the algebraic dual, whereas we adopt the more usual notation where X^* denotes the topological dual.

> distribution that is used in the Central Limit Theorems.) Remarkably, in a dual pair $\langle X, X^* \rangle$, any subspace of X^* that separates the points of X is weak-* dense in X^*.
>
> Debreu (1954) introduced dual pairs in economics in order to describe the duality between commodities and prices. According to this interpretation, a dual pair $\langle X, X^* \rangle$ represents the commodity-price duality, where X is the commodity space, X^* is the price space, and $\langle x, x^* \rangle$ is the value of the bundle x at prices x^*. ...

Again, on page 163, they refer to "the remarkable Alaoglu Theorem" that asserts that the unit ball in X^* is compact in the weak-* topology.

In this paper, we adopt the duality of functions and measures and apply it to the duality of prices and quantities; however we find it convenient to reverse the roles of the dual pair $\langle X, X^* \rangle$, letting the space of functions X represent the price space and the space of measures X^* represent the commodity space. We equip X^* with the weak-* topology. This has the effect (among others) of making X the topological dual of X^*.

References

Aliprantis, C. D. and K. C. Border (1999). *Infinite dimensional analysis: A hitchhiker's guide* (2nd ed.). Springer.

Back, K. and S. R. Pliska (1991). On the fundamental theorem of asset pricing with an infinite state space. *Journal of Mathematical Economics 20*, 1–18.

Bewley, T. (1972). Existence of equilibria in economies with infinitely many commodities. *Journal of Economic Theory 4*, 514–540.

Clark, S. A. (1993). The valuation problem in arbitrage price theory. *Journal of Mathematical Economics 22*, 463–478.

Clark, S. A. (2000). Arbitrage approximation theory. *Journal of Mathematical Economics 33*, 167–181.

Clark, S. A. (2002). Arbitrage on a lattice of derivative securities. Working paper, University of Kentucky.

Debreu, G. (1954). Valuation, equilibrium, and Pareto optimum. *Proceedings of the National Academy of Sciences 40*, 588–592.

Delbaen, F. and W. Schachermayer (1994). A general version of the fundamental theorem of asset pricing. *Mathematische Annalen 300*, 463–520.

Doob, J. L. (1994). *Measure Theory.* Springer-Verlag.

Gilles, C. and S. F. LeRoy (1997). Bubbles as payoffs at infinity. *Economic Theory 9*, 261–281.

Gilles, C. and S. F. LeRoy (1998). Arbitrage, martingales, and bubbles. *Economic Letters 60*, 357–362.

Harrison, J. M. and D. M. Kreps (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory 20*, 381–408.

Jones, L. E. (1984). A competitive model of commodity differentiation. *Econometrica 52*(2), 507–530.

Kreps, D. M. (1981). Arbitrage and equilibrium in economies with infinitely many commodities. *Journal of Mathematical Economics 8*, 15–35.

Mas-Colell, A. (1975). A model of equilibrium with differentiated commodities. *Journal of Mathematical Economics 2*, 263–295.

Pliska, S. R. (1997). *Introduction to mathematical finance: Discrete time models.* Blackwell Publishers Inc.

Werner, J. (1997). Arbitrage, bubbles, and valuation. *International Economic Review 38*(2), 453–464.

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