BUBBLES AND ARBITRAGE

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ABSTRACT. The payout for doubling strategy contains a bubble component in the sense of Gilles and LeRoy (1997). To support the claim that the doubling strategy is an arbitrage, the standard analysis relies on discontinuities in marginal utility (*i.e.,* value) that arise from arbitrarily assigning zero value in the direction of the bubble component. However, the doubling strategy does not present an arbitrage opportunity for an agent with continuous marginal utility because the payout (including the negative bubble component) is not in the positive orthant of the appropriate space. By removing the discontinuities from the valuation operator, standard absence-of-arbitrage arguments can be applied to L^1-bounded local martingales. Consequently, once an agent with continuous marginal utility is admitted, there are no arbitrages in the standard continuous-time finance model.

1. INTRODUCTION

The doubling strategy is the bugbear of continuous-time finance. It is a self-financing trading strategy that generates something for nothing—an arbitrage. Any agent who prefers more to less has an unbounded demand for such a trading strategy. Even the representative agent wants to trade: Robinson Crusoe, endowed only with his coconut tree, yearns to be confronted with a price system proportional to his marginal utility and bankrolled with wealth equal to the value of his tree so that he can undertake the doubling strategy. The response to this intolerable state of affairs varies across jurisdictions: Some limit the amount a trader may borrow while others outlaw the trading strategies that require more than limited borrowing. Our response is different. We confirm that Crusoe does indeed prefer more to less, and then we show that Crusoe's marginal utility in the direction of the doubling strategy is zero. Therefore, the doubling strategy is simply not an arbitrage. The price system constructed from Crusoe's marginal utility is in fact a no-trade price system. No restrictions on trade are required.

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The source of the apparent arbitrage is the discontinuity of marginal utility that is arbitrarily and unnecessarily imposed. Continuity is the purview of topology, so we seek a topological vector space for which marginal utility (*i.e.,* value) is continuous.

The discontinuity arises because the sequence of payouts that comprise the doubling strategy converges pointwise but not uniformly. It is rather like the Three Stooges' routine in which Moe directs Curly to close all the drawers in a chest of drawers, but each time Curly closes one drawer another pops open. For our purposes, suppose the chest had an infinite number of drawers, all closed but the first, and Curly systematically worked his way through the drawers, closing drawer after drawer with the predicable result that each time he closed one the next one in order popped open. When questioned by Moe, Curly would explain with satisfaction that every drawer is eventually permanently closed, to which an exasperated Moe would respond that there was always one drawer open ("you knucklehead!"). Larry would try to mediate the disagreement, observing that Curly had a pointwise notion of completing the task, while Moe had a uniform notion, at which point Moe would knock Curly's and Larry's heads together.

Suppose each drawer is associated with a specific commodity.[1] An open drawer represents the presence of one unit of that commodity in a bundle of commodities. In general, many drawers could be open simultaneously. Curly, however, generates a specific sequence of bundles $\langle x_n \rangle$, where each bundle is itself an infinite sequence,

$$x_1 = \{1, 0, 0, 0, \cdots\}$$
$$x_2 = \{0, 1, 0, 0, \cdots\}$$
$$x_3 = \{0, 0, 1, 0, \cdots\}$$

and so forth. This sequence of bundles converges pointwise to a bundle represented by a sequence of zeros, but it does not converge uniformly. Now suppose Crusoe gets one unit of marginal utility from each good in the bundle. Each of the bundles that Curly generates produces one unit of marginal utility for Crusoe, so of course the sequence of marginal utilities converges to one.

We can now pose the central question: What is the marginal utility of the limiting bundle? Is it zero (in the limit every good is absent from the bundle) or is it one (every bundle in the sequence has one item in it)? The standard practice is to insist that every rational agent must agree that the marginal utility of the limiting bundle is zero. However, our man Crusoe demurs. He claims the marginal utility of the limiting bundle for him is one. The resident psychiatrist is willing to declare Crusoe rational, but only if there is an objective bundle to which such marginal utility can be ascribed. But in what sense can the sequence of bundles have converged if not pointwise or uniformly? As it turns out, the sequence of bundles has limit points

[1]This example is corresponds to the treatment of the very-long discount bond in the appendix in Gilles and LeRoy (1997). In their setting, the sale of such a bond is equivalent to a Ponzi scheme in which an investors rolls over short-term borrowing forever. In fact, our critique of the standard treatment of the doubling strategy applies equally well to the standard treatment of Ponzi schemes: The assertion that a Ponzi scheme constitutes an arbitrage opportunity relies on discontinuities in marginal utility. In a future revision, we will emphasize the similarity between the two issues in the body of paper.

in a larger space equipped with the weak* topology; however, unlike the bundles in the sequence, the limit-point bundles cannot be described by itemizing their contents. Nevertheless, these limit points are well-defined objects, any of which provides Crusoe with marginal utility of one.

More formally, adopting either almost sure convergence or convergence in measure for sequences in L^1 leads to discontinuous valuation operators that produce the appearance of arbitrage opportunities. These arbitrages are shown to be nonexistent when one adopts instead weak* convergence in $(L^1)^{**}$, the bi-dual of L^1, for which the valuation operator is continuous. The classic example is the doubling strategy, which amounts to a sequence of random variables that converges in probability, but does not converge in the L^1 norm topology. As a result, the value (*i.e.,* expectation) of the probability limit (which is positive) does not equal the limit of the values of the sequence (which is zero)—hence, a discontinuity. Nevertheless, the sequence has weak* limit points in $(L^1)^{**}$, all of which have value zero (the limit of the sequence of values). In other words, in $(L^1)^{**}$ equipped with the weak* topology, valuation is continuous and consequently the doubling strategy is not an arbitrage.[2]

The discontinuities in valuation (resulting from the standard convergence criteria) reflect the existence of payout bubbles in the sense of Gilles and LeRoy (1997), where payouts are understood as elements of $(L^\infty)^* = (L^1)^{**} \supset L^1$. In that paper, the price system is in L^∞ and payouts are in the dual space $(L^\infty)^*$.[3] Such a payout can be uniquely decomposed into a fundamental component in L^1 (the measure limit) and a bubble component in the orthogonal complement of L^1. The fundamental component can be identified with a signed measure (a countably-additive set function), while the bubble component can be identified with a pure charge (a finitely-additive set function).[4]

In present paper, we take as given an economy in which the valuation of (fundamental) securities is computed by integration with respect to some measure μ. Thus, we require *deflated payouts* (*i.e.,* prices times payouts) to be in $L^1(\mu)$. We then change to an equivalent measure ψ (constructing the Radon–Nikodym derivative from the price system) where the payouts are themselves in $L^1(\psi)$. At this point the analysis can proceed as outlined above.[5] For example, we can apply the analysis of bubbles-as-charges to the standard continuous-time finance model

[2]Kreps (1981), in his seminal discussion of arbitrage in economies with infinitely many commodities, lists criteria for preferences that are suitable to support no-arbitrage price equilibria. Importantly, these criteria include the continuity of preferences in the topology adopted. This criterion is violated by the standard analysis, which implicitly adopts the topology associated with convergence in measure.

[3]The roles of prices and quantities are reversed from their earlier paper, Gilles and LeRoy (1992), where payouts are in L^∞, price systems in $(L^\infty)^*$, and bubbles are embedded in the price system. Fairly mild restrictions on preferences and opportunity sets rule out bubbles in the price system. See Stokey, Lucas, and Prescott (1989, Chapter 15) and Back and Pliska (1991).

[4]It should be noted that Gilles and LeRoy (1997) discuss the doubling strategy and we are entirely in agreement with their analysis. Our contribution on this front is to flesh out the analysis and extend its applicability to continuous-time finance in particular.

[5]Gilles and LeRoy (1997), in their analysis of the Miller–Modigliani example in which the fundamental payouts are not integrable with respect to Lebesgue measure, use the price system

in which the price system and a finite set of basis securities are all in $L^2(P)$ and deflated fundamental payouts are in $L^1(P)$, where $\mu = P$ is the physical probability measure. After a change to an equivalent probability measure Q, self-financing trading strategies generate local martingales. The theory of charges is applicable to $L^1(Q)$-bounded local martingales, which includes not only the doubling strategy but also (among other things) positive local martingales. The value of the doubling strategy is zero because the negative value of the bubble component exactly offsets the positive value of the fundamental component.

We have one further introductory observation. One may have no compunction outlawing the doubling strategy, a peculiar form of gambling that appears to have little connection with allocational efficiency. Why would anyone ever want to insure against the event "a fair coin never comes up heads"? The very question, however, raises the issue of the interpretation and treatment of Arrow–Debreu securities—that is, securities that pay "one unit in one state of the world." When there are an uncountable number of states, Arrow–Debreu securities have positive prices but zero expected payouts. They are pure bubbles. If these securities are arbitrages, they must be outlawed—as of course they are in standard practice—a sorry conceptual state of affairs indeed.[6]

Literature review. There is a large and growing literature on rational bubbles and bubble-related issues. In a later version, we will include a more substantial literature review. In the meantime, this must suffice.

We were inspired in part by LeRoy (2000), who addresses (in a somewhat different framework) the central idea—that adopting the weak* topology in $(L^\infty)^*$ eliminates the valuation discontinuities inherent in pointwise convergence and thereby simplifies the framework for asset pricing.

Our paper has relevance for Loewenstein and Willard (2000). What they identify as the bubble component of an asset's value (p. 29) agrees with our analysis—it is the value of the part of the payout in the orthogonal component of L^1, which is the value derived from a pure charge. However, because they do not recognize the payout that supports the bubble value,[7] they treat the presence of a bubble component as an arbitrage opportunity, a treatment that is fully consistent with the standard treatment of the doubling strategy, to which of course we take exception.

2. Arbitrage and value: The main point in a nutshell

We begin with a standard definition of an arbitrage and go from there. Given a Banach lattice X (a normed linear vector space equipped with a vector order $\geq$),

(in effect) to redefine the norm in order to ensure the sequence of fundamental payouts is in an L^1 space.

[6]Consequently, "complete markets" are characterized in terms of random variables with finite variance (payoffs in L^2) rather than in terms of Arrow–Debreu securities. As it turns out, the martingale representation theorem can be used to show that a market that is "L^2-complete" would be "Arrow–Debreu complete" if the trading restrictions of standard practice were removed.

[7]We would amend their description (p. 29) of the bubble value as follows: "A bubble therefore represents the amount by which the equilibrium price of an asset exceeds the present value of its [*fundamental*] payouts."

define the positive cone $K = X_+ \setminus \{0\}$, where the origin has been removed.[8] A security's payout ζ is an element of X. Assume a mapping $h : X \to \mathbb{R}$ is given, and let $h(\zeta)$ denote the cost of ζ. In general, there is no requirement for h to be positive, linear, or continuous. A payout ζ_0 is an arbitrage if $\zeta_0 \in K$ and $h(\zeta_0) \leq 0$.

A valuation operator is a positive bounded linear functional $\mathcal{V} \in X^*$, where X^* is the normed dual space of X and $\mathcal{V} : X \to \mathbb{R}$. The positivity of $\mathcal{V}$ is expressed by $\zeta \in K \implies \mathcal{V}(\zeta) > 0$. Obviously, a payout ζ_0 is not an arbitrage if $h(\zeta_0) = \mathcal{V}(\zeta_0)$.

Fix a measure space $(\Omega, \mathcal{F}, \psi)$, where $\mathcal{F}$ is ψ-complete and ψ is nonnegative, countably additive, and σ-finite.[9] The space of payouts is $X = (L^1)^{**}$, the bidual of L^1. (Note $L^1 \subset (L^1)^{**}$.) The positive cone is $K = (L^1)^{**}_+ \setminus \{0\}$. Let the valuation operator be the "unit functional" $\mathbf{1}$, which equals the number one almost everywhere. (Note $\mathbf{1} \in L^\infty = (L^1)^* \subset (L^1)^{***} = X^*$.)[10]

Let $\langle z_n \rangle$ be a sequence where $z_n \in L^1$ for all n. Let $\mathrm{Lim}^*\langle z_n \rangle$ denote the set of weak* limit points of the sequence, where $\mathrm{Lim}^*\langle z_n \rangle \subset (L^1)^{**}$.[11] If the sequence has no limit points, then $\mathrm{Lim}^*\langle z_n \rangle = \varnothing$.

Definition 1. *If any $\zeta \in \mathrm{Lim}^*\langle z_n \rangle$ is an arbitrage, then we say the sequence $\langle z_n \rangle$ generates an arbitrage; otherwise, we say the sequence does not generate an arbitrage.*

If $h(\zeta) = \mathbf{1}(\zeta)$ for every $\zeta \in \mathrm{Lim}^*\langle z_n \rangle$, then the sequence $\langle z_n \rangle$ does not generate an arbitrage. In particular, consider the following case. If $v = \lim_{n\to\infty} \mathbf{1}(z_n)$ exists and is finite, then (by the definition of weak* convergence) $\mathbf{1}(\zeta) = v$ for every $\zeta \in \mathrm{Lim}^*\langle z_n \rangle$. If, in addition, $h(\zeta) = v$ for every $\zeta \in \mathrm{Lim}^*\langle z_n \rangle$, then the sequence does not generate an arbitrage. Specializing this result a bit, if $z_0 = \mathbf{1}(z_n)$ for all n and $h(\zeta) = z_0$ for all $\zeta \in \mathrm{Lim}^*\langle z_n \rangle$, then the sequence does not generate an arbitrage.

Usage. As an example, we apply our setup to continuous-time finance and show that no arbitrage opportunities given a state–price deflator.[12] For simplicity, assume a

[8] We adopt the following sign conventions (where the inequalities are understood to hold almost everywhere): $x \geq 0$ means x is nonnegative; $x > 0$ means x is nonnegative and not zero, but not necessarily strictly positive in all coordinates; and $x \gg 0$ means x is strictly positive.

[9] Given a measure space $(\Omega, \mathcal{F}, \mu)$, recall that $L^p \triangleq L^p(\mu) \triangleq L^p(\Omega, \mathcal{F}, \mu)$ is the space of measurable functions f on Ω for which $\int_\Omega |f|^p \, d\mu < \infty$ for $1 \leq p < \infty$ and for essentially bounded f for $p = \infty$. The norm is defined as $\|f\|_p \triangleq \|f\|_p^\mu \triangleq \left(\int_\Omega |f|^p \, d\mu\right)^{1/p}$ for $1 \leq p < \infty$ and as the essential supremum of $|f|$ for $p = \infty$. Two functions that agree a.e. (almost everywhere) are identified. A sequence $\langle f_n \rangle$ is said to be L^p-bounded if $\sup_{n \in \mathbb{N}} \|f_n\|_p < \infty$.

[10] For $z \in L^1$, $\mathbf{1}(z) = \int_\Omega z \, d\psi$. More generally, for $\zeta \in (L^1)^{**}$, $\mathbf{1}(\zeta) = \mathbf{1}(\widetilde{z}) + \int_\Omega d\varphi$, where $\widetilde{z} \in L^1$ and $\varphi \in \mathbf{pch}$ is a pure charge. We refer to $\widetilde{z}$ as the fundamental component of the payout ζ and to φ as the bubble component. (The decomposition is unique.) Correspondingly, we refer to $\mathbf{1}(\widetilde{z})$ and the fundamental value and $\int_\Omega d\varphi$ as the bubble value.

[11] A sequence generates a set of weak* limit points via its convergent subnets. Nets may be thought of as generalizations of sequences, which are insufficient for characterizing weak* convergence in $(L^1)^{**}$. A net involves a partial order on a set according to which, given any two elements of the set, there always exists an element "greater than" both. Just as sequences have subsequences, nets have subnets. Since a sequence is a net, sequences have subnets.

[12] We flesh out this example below.

finite trading interval $\mathcal{T} = [0, T]$. Given a state–price deflator, a self-financing trading strategy can be expressed as a local martingale M, where the cost of the trading strategy is $M(0)$. By the definition of a local martingale, M can be reduced via a fundamental sequence of stopping times $\langle \tau_n \rangle$ to a sequence of martingales $\langle M_n \rangle$ for which $E[M_n(T)] = M(0)$ for all n. To apply the preceding setup, let $z_0 = M(0)$, $z_n = M_n(T)$, and $\mathbf{1}(\,\cdot\,) = E[\,\cdot\,]$. Consequently, no self-financing trading strategy (including the doubling strategy) generates an arbitrage.[13]

3. Discussion

In the previous section, the measure space $(\Omega, \mathcal{F}, \psi)$ used to characterize asset-pricing is taken as given. In practice, what is often given is a related measure space $(\Omega, \mathcal{F}, \mu)$ where μ is equivalent to ψ and is considered the *natural* measure. For our purposes, let μ be nonnegative, countably additive, and σ-finite, and let $\mathcal{F}$ be μ-complete. Let $\pi \gg 0$ be a price system (*i.e.,* the state prices), z_n be an asset's payout (*i.e.,* the quantities in each state), and $\pi\, z_n$ be the deflated payout (*i.e.,* the value of the asset on a state-by-state basis). Assume $\pi\, z_n \in L^1(\mu)$. The value v_n of the asset is the state-by-state weighted average of its deflated payouts: $v_n = \int_\Omega \pi\, z_n\, d\mu$. We can express an asset's value in terms of an equivalent measure ψ: $v_n = \int_\Omega z_n\, d\psi$, where $z_n \in L^1(\psi)$ and $\frac{d\psi}{d\mu} = \pi$ is the Radon–Nikodym derivative. If $\pi \in L^1(\mu)$, then ψ is finite. Define the valuation operator $\mathcal{V}[z] = \int_\Omega z\, d\psi$ for $z \in L^1(\psi)$.[14]

Value and marginal utility. In order to illustrate the connection between value and marginal utility, consider the following class of utility functions:

$$U(c) \triangleq \int_\Omega u(c(\omega), \omega)\, \mu(d\omega) \quad \text{where } u_c(c, \omega) \triangleq \frac{\partial u(c, \omega)}{\partial c} \gg 0.$$

Letting $d\psi^c/d\mu = u_c(c, \,\cdot\,)$, we can express utility as $U(c) = \int_\Omega \frac{u(c(\omega),\omega)}{u_c(c(\omega),\omega)}\, \psi^c(d\omega)$. Here $U : \mathcal{C} \to \mathbb{R}$, where $\mathcal{C} = \{c : u(c, \,\cdot\,)/u_c(c, \,\cdot\,) \in L^1(\psi^c)\}$. Given an endowment c, define the set of feasible directions:

$$F(c) = \{z \in L^1(\psi^c) : \exists\, \epsilon \in (0, 1),\;\; c + \alpha\, z \in \mathcal{C},\;\; \alpha \in [0, \epsilon]\}.$$

We assume $z > 0$ is feasible. Fixing c, we compute the utility gradient in the direction of $z \in F(c)$ by the Gateaux derivative:

$$\nabla U(c; z) = \lim_{\alpha \downarrow 0} \frac{U(c + \alpha\, z) - U(c)}{\alpha} = \int_\Omega u_c(c(\omega), \omega)\, z(\omega) \mu(d\omega) = \int_\Omega z\, d\psi^c,$$

where $\nabla U(c; \,\cdot\,) \in L^1(\psi^c)^*$. We can identify $\nabla U(c; \,\cdot\,)$ with $\mathcal{V}[\,\cdot\,]$. "More is preferred to less" is expressed by $z > 0 \implies \nabla U(c; z) = \mathcal{V}[z] > 0$.

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[13]Local martingales whose martingale sequences have no weak* limit points are not in $(L^1)^{**}$. We do attempt to characterize trading strategies that produce payouts in $(L^1)^{**}$.

[14]In some cases, it is convenient to compute the Radon–Nikodym derivative as $\frac{d\psi}{d\mu} = \beta\, \pi$ for some payoff $\beta \gg 0$ for which $\int_\Omega \beta\, \pi\, d\mu = 1$.

Extending the valuation operator. Let $X = L^1(\Omega, \mathcal{F}, \psi)$ and let $f \in X^*$.[15] Any functional $f : X \to \mathbb{R}$ can be immediately extended to $f : X^{**} \to \mathbb{R}$ (by virtue of the fact that $f \in X^{***}$). Let $\mathbf{1}$ denote the unit functional in X^*. By the Riesz Representation Theorem, we can express the valuation operator as $\mathcal{V}[z] = \mathbf{1}(z)$ for $z \in X$, which can be extended to $\mathcal{V}[w] = \mathbf{1}(w)$ for $w \in X^{**}$.[16]

In general, $\mathbf{1}(w)$ cannot be represented in terms of an integral with respect to ψ. In order to obtain a representation for $\mathbf{1}(w)$, we turn to the space of charges.[17] Let $\mathbf{ba}(\Omega, \mathcal{F}, \psi)$ denote the space of bounded charges on $\mathcal{F}$ that vanish on sets of ψ-measure zero. Charges are finitely-additive set functions (generalizations of signed measures). Let $\mathbf{ca}(\Omega, \mathcal{F}, \psi)$ denote the space of bounded signed measures on $\mathcal{F}$ that vanish on sets of ψ-measure zero. Note that $\mathbf{ca} \subset \mathbf{ba}$. Let $\mathbf{pch}(\Omega, \mathcal{F}, \psi)$ denote the space of *pure charges* on $\mathcal{F}$. The spaces $\mathbf{ca}$ and $\mathbf{pch}$ are orthogonal complements: For any $\xi \in \mathbf{ba}$, there is a unique decomposition $\xi = \rho + \varphi$, where $\rho \in \mathbf{ca}$ and $\varphi \in \mathbf{pch}$. There are isometric-isomorphisms between $w \in X^{**}$ and $\xi \in \mathbf{ba}$ on the one hand and $\widetilde{z} \in X$ and $\rho \in \mathbf{ca}$ on the other, expressed by the identities $w(f) = \int_\Omega f \, d\xi$ and $\rho[A] = \int_A \widetilde{z} \, d\psi$, where $f \in X^*$ and $A \in \mathcal{F}$. In addition, we have $w(f) = f(w)$. These relations allow us to write

$$\mathbf{1}(w) = \int_\Omega \widetilde{z} \, d\psi + \int_\Omega d\varphi, \tag{3.1}$$

where $\widetilde{z} \in X$ is the fundamental component of w and φ is a pure charge.[18] We can write (3.1) as $\mathcal{V}[w] = \mathcal{V}[\widetilde{z}] + \varphi[\Omega]$, where $\mathcal{V}[\widetilde{z}]$ is the value of the fundamental component (the fundamental value) and $\varphi[\Omega]$ is the value of the bubble component (the bubble value).

Using the theory of charges, we prove[19] that if a net converges in the weak* topology to $w \in (L^1)^{**}$, then it converges in measure (and a subnet converges almost surely) to $\widetilde{z} \in L^1$, where $\widetilde{z}$ is the fundamental component of w.

The typical setting. In our applications, we make the following standing assumptions: The sequence $\langle z_n \rangle$ is $L^1(\psi)$-bounded and converges to $\widetilde{z}$ either almost surely or in ψ-measure and $v = \lim_{n\to\infty} \mathcal{V}[z_n]$ exists. Given Theorems A.1–A.10 in the Appendix, we know the sequence has weak* limit points in $L^1(\psi)^{**}$, the value of the limit points is v, the fundamental component is $\widetilde{z}$, the value of the fundamental component is $\mathcal{V}[\widetilde{z}]$, and the value of the bubble component is $v - \mathcal{V}[\widetilde{z}]$. In addition, if the sequence converges in the norm topology, then the bubble component is zero and $v = \mathcal{V}[\widetilde{z}]$.

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[15] X^* denotes the topological dual space of X; *i.e.,* X^* is the space of bounded linear functionals on X. Any normed linear space X can be naturally embedded in its bi-dual X^{**}; *i.e.,* $X \subseteq X^{**}$.

[16] The formal extension of the utility gradient is identical to the extension of the valuation operator. In principle, an extended utility function can be obtained by integration from the extended gradient. However, given the gradient, the utility function itself plays no role in the sequel.

[17] See Appendix A for the basic theorems.

[18] Given Theorems A.1–A.4 in the Appendix, $\mathcal{V}[w] = \mathbf{1}(w) = w(\mathbf{1}) = \int_\Omega d\xi = \int_\Omega d\rho + \int_\Omega d\varphi$, where $\int_\Omega d\rho = \rho[\Omega] = \int_\Omega \widetilde{z} \, d\psi = \mathbf{1}(z) = \mathcal{V}[\widetilde{z}]$ and $\int_\Omega d\varphi = \varphi[\Omega]$.

[19] See Theorem A.5 in the Appendix.

Arrow–Debreu securities. Suppose Ω is a subspace of $\mathbb{R}^d$ and let Ψ be the (non-decreasing) generating function for the Stieltjes measure ψ. In addition, suppose $\Psi = \Psi_c + \Psi_d$, where Ψ_c is an absolutely continuous function generating an absolutely continuous measure ψ_c and Ψ_d is a jump function generating a discrete measure ψ_d.[20] Then

$$v_n = \int_\Omega z_n \, d\psi = \int_\Omega z_n(\omega) \, d\Psi(\omega) = \int_\Omega z_n(\omega) \, \partial_\omega \Psi_c(\omega) \, d\omega + \sum_{i=1}^m z_n(\omega_i) \, h_i,$$

where $\partial_\omega \Psi_c$ is the density (with respect to Lebesgue measure) of ψ_c and $\{h_i\}_{i=1}^m$ are jumps in Ψ_d that correspond to the discontinuity points $D = \{\omega_i\}_{i=1}^m$.

For $\omega_i \in D$, it is straightforward to identify the value of an Arrow–Debreu security with h_i. Let[21] $z_n(\omega) = 1_{\{\omega_i\}}(\omega)$, so that $z_n(\omega)$ pays one unit in state ω_i (and zero elsewhere). Clearly $v_n = h_i$.

If the density is continuous at $\omega_0 \in \Omega \setminus D$, we can identify $\partial_\omega \Psi_c(\omega_0)$ with the value of an Arrow–Debreu security for that state of the world. For simplicity let $\Omega = \mathbb{R}$ and $D = \varnothing$, and assume $\Psi_c(\omega)$ is strictly increasing in ω. Let $z_n(\omega) = (n/2)\, 1_{\{\omega_0 - 1/n \le \omega \le \omega_0 + 1/n\}}(\omega)$. In this case, $z_n(\omega)$ pays a total of one unit centered on ω_0: $\int_\Omega z_n(\omega)\, d\omega = (n/2) \int_{\omega_0 - 1/n}^{\omega_0 + 1/n} d\omega = 1$. The payment becomes more and more concentrated on ω_0 as $n \to \infty$. The limiting payout is an Arrow–Debreu security.[22] The payout converges to zero almost surely: $\lim_{n\to\infty} \; z_n(\omega) = 0$ for all $\omega \in \Omega \setminus \omega_0$. The value of z_n is $v_n = (n/2) \int_{\omega_0 - 1/n}^{\omega_0 + 1/n} \partial_\omega \Psi_c(\omega)\, d\omega$ and $\lim_{n\to\infty} \; v_n = \partial_\omega \Psi_c(\omega_0)$. Since $z_n \ge 0$, $\|z_n\|_1 = v_n$, and so $\langle z_n \rangle$ is $L^1(\psi)$-bounded. The sequence does not converge in norm, since $\lim_{n\to\infty} \; \|z_n - 0\|_1 = v > 0$. Therefore, the Arrow–Debreu security is a pure bubble with value $\partial_\omega \Psi_c(\omega_0)$.

Convergence in measure and convergence in norm. Here we investigate the relation between convergence in measure and convergence in norm. We assume ψ is finite, in which case almost sure convergence implies convergence in measure. Given convergence in measure and the $L^1(\psi)$-boundedness of $\langle z_n \rangle$, the following condition is necessary and sufficient for convergence in norm.[23]

Condition 1. *For any given $\varepsilon > 0$, there exists $\delta > 0$ such that for each $A \in \mathcal{F}$ with $\psi[A] \le \delta$ we have* $\sup_n \; \int_A |z_n| \, d\psi \le \varepsilon$.

Given our standing assumptions, Condition 1 is necessary and sufficient for the bubble component of the limit points to be zero. If Condition 1 is false, then the sequence does not converge in the norm topology and $v \ne \int_\Omega \widetilde{z} \, d\psi$.

[20]See Kolomogorov and Fomin (1970) for a discussion of Stieltjes measures.

[21]The indicator (or characteristic) function is defined as

$$1_A(x) \triangleq \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

[22]The reader may recognize the limiting payout as a Dirac delta function.

[23]An L^1-bounded sequence that satisfies Condition 1 is called *uniformly integrable*.

The relation between the bubble value and the failure to converge in the L^1 norm topology can be illustrated simply. Define the norm-divergence,

$$d \triangleq \lim_{n\to\infty} \int_\Omega |z_n - \widetilde{z}|\, d\psi,$$

if the limit exists. By definition, the sequence converges to $\widetilde{z}$ if and only if $d = 0$. If the sequence is nonnegative and if $\widetilde{z} = 0$, then $d = v$; in other words, the norm-divergence equals the bubble value.

In order to investigate the wedge between convergence in measure and convergence in norm in more detail, define the set $A_{n,\varepsilon} \triangleq \{\omega : |z_n(\omega) - \widetilde{z}(\omega)| > \varepsilon\}$, which is the support for where the divergence between z_n and $\widetilde{z}$ is greater than ε. The measure of this support is $\psi[A_{n,\varepsilon}] = \int_{A_{n,\varepsilon}} d\psi$. Convergence in measure requires only that for every $\varepsilon > 0$, $\lim_{n\to\infty} \psi[A_{n,\varepsilon}] = 0$. In other words, the support of the divergence between z_n and $\widetilde{z}$ must go to zero, but the divergence itself plays no role. By contrast, the support and the divergence play more equal roles in the sequence involved in computing norm convergence. In particular, we can use $A_{n,\varepsilon}$ to partition Ω and see how the failure to converge in the norm is related to the bubble value. First, by the definition of convergence in measure,

$$\lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} \int_{\Omega\setminus A_{n,\varepsilon}} |z_n - \widetilde{z}|\, d\psi = 0.$$

Therefore,

$$\lim_{n\to\infty} \int_\Omega |z_n - \widetilde{z}|\, d\psi = \lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} \int_{A_{n,\varepsilon}} |z_n - \widetilde{z}|\, d\psi.$$

We can compute value with respect to the same partition:

$$\begin{aligned} v = \lim_{n\to\infty} \int_\Omega z_n\, d\psi &= \left(\lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} \int_{\Omega\setminus A_{n,\varepsilon}} z_n\, d\psi\right) + \left(\lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} \int_{A_{n,\varepsilon}} z_n\, d\psi\right) \\ &= \int_\Omega \widetilde{z}\, d\psi + \left(\lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} \int_{A_{n,\varepsilon}} z_n\, d\psi\right). \end{aligned} \tag{3.2}$$

The second term in the second line of (3.2) provides an explicit direct construction of the bubble value. In practice, of course, one simply computes $v - \mathcal{V}[\widetilde{z}]$.

4. Miller–Modigliani example

Gilles and LeRoy (1997) present this as the archetypical example of a payout bubble, where the value is attributable to payouts that occur after every finite time. We treat the example as an introduction to our approach.

The setup. Take as given the measure space $(\mathbb{R}_+, \mathfrak{B}(\mathbb{R}_+), \mathrm{Leb})$, where $\mathbb{R}_+ = [0, \infty]$, $\mathfrak{B}(\mathbb{R}_+)$ is the set of Borel sets generated by $\mathbb{R}_+$, and Leb is Lebesgue measure. Also take as given the following utility function: $U(c) = \int_0^\infty e^{-r\,\omega} \log(c(\omega))\, d\omega$, where $r > 0$ is the rate of time preference. Marginal utility is given by $\nabla U(c; z) = \int_0^\infty e^{-r\,\omega}\, c(\omega)^{-1}\, z(\omega)\, d\omega$. Let the endowment be $c(\omega) \equiv 1$, so that $\pi(\omega) = e^{-r\,\omega}$ is the price system. Since $\|\pi\|_1^\mu = \int_0^\infty e^{-r\,\omega}\, d\omega = 1/r < \infty$, the measure ψ is finite (where the Radon–Nikodym is given by $\frac{d\psi}{d\mathrm{Leb}} = e^{-r\,\omega}$). In this setting we have

$v_n = \int_0^\infty z_n(\omega)\,\partial_\omega \Psi_c(\omega)\,d\omega$, where $\partial_\omega \Psi_c(\omega) = e^{-r\,\omega}$. Therefore, we can identify zero-coupon bonds with Arrow–Debreu securities, which are pure bubbles.

The example. Let $s(\omega)$ be the amount of capital a firm has at time ω and let the net earnings of the firm be $r\,s(\omega)$. If a firm pays dividends at the rate $z(\omega) = \gamma\,r\,s(\omega)$ with dividend–payout ratio $\gamma \geq 0$, then the change in the firm's capital is $s'(\omega) = (1-\gamma)\,r\,s(\omega)$, which is an ordinary differential equation. For a firm that has one unit of capital at time zero, the solution to this equation is $s(\omega) = e^{r\,(1-\gamma)\,\omega}$, and the dividend stream generated by this firm is $z(\omega) = \gamma\,r\,e^{r\,(1-\gamma)\,\omega}$. For $\gamma > 0$, the present value of the dividends is $\int_0^\infty z(\omega)\,\pi(\omega)\,d\omega = \int_0^\infty r\,\gamma\,e^{-r\,\gamma\,\omega}\,d\omega = 1$. The value attributable to dividends paid after $\omega = T$ is $\int_T^\infty z(\omega)\,\pi(\omega)\,d\omega = e^{-r\,\gamma\,T}$. Thus, as $\gamma \downarrow 0$ the value attributable to dividends paid beyond any finite time goes to one.

Now consider a sequence of dividend streams $\langle z_n \rangle$, where $z_n(\omega) = r\,\gamma_n\,e^{r\,(1-\gamma_n)\,\omega}$ and where $\langle \gamma_n \rangle$ is any positive, monotonically decreasing sequence that converges to zero. Obviously, $\lim_{n\to\infty}\; v_n = 1$. The reader may confirm that $\langle z_n \rangle$ is $L^1(\psi)$-bounded, converges almost surely to zero, and does not converge in norm. Consequently, the value (*i.e.,* marginal utility) of the limiting dividend stream is one; the fundamental value is zero and the bubble value is one.

The Very-long discount (VLD) bond. Before leaving this economy, we examine the value of an infinite-horizon zero-coupon bond. Let

$$z_n(\omega) = \left(\frac{r\,e^{r\,n}}{1 - e^{-r/n}}\right) 1_{\{n \leq \omega \leq n+1/n\}}(\omega).$$

In this example, the payout is a deferred annuity that pays at the fixed rate of $(r\,e^{r\,n})/(1 - e^{-r/n})$ from $\omega = n$ to $\omega = n + 1/n$. Thus n controls the starting date, the flow duration, and the flow rate. The total flow grows with n and is unbounded as $n \to \infty$. Once again $v_n = 1$ and again the reader may confirm the sequence is $L^1(\psi)$-bounded, converges almost surely to zero, and does not converge in norm. Consequently, the value of the limiting payout is one; the fundamental value is zero and the bubble value is one.

The VLD bond in a discrete-time economy. The chest-of-drawers illustration (in the Introduction) is based on this example, which (as noted above) is treated in the appendix of Gilles and LeRoy (1997). Let $\Omega = \mathbb{N}$ be the natural numbers, $\mathcal{F} = \mathcal{P}(\mathbb{N})$ be the set of all subsets of $\mathbb{N}$, and $\mu = \nu$ be the counting measure (*i.e.,* $\nu[A] = \sum_{\omega\in A} 1$ is the number of elements of in A). Note that ν is σ-finite but not finite. Given this setup, $L^1(\mu) = \ell^1(\nu)$ is the space of absolutely summable sequences. Let $\pi(\omega) = (1+r)^{-\omega}$, where $r \geq 0$. The Arrow–Debreu securities are fundamental in this case. Let $z_n(\omega) = (1+r)^\omega\,1_{\{\omega=n\}}(\omega)$, so that $\pi(\omega)\,z_n(\omega) = 1_{\{\omega=n\}}(\omega)$ and $\pi\,z_n \in \ell^1(\nu)$ for all $r \geq 0$. The value of z_n is $v_n = \sum_{\omega=1}^\infty 1_{\{\omega=n\}}(\omega) = 1$. Using π as the Radon–Nikodym derivative, $\psi[A] = \sum_{\omega\in A}(1+r)^{-\omega}$. If $r > 0$, then $\pi \in \ell^1(\nu)$, and $\psi[\mathbb{N}] = 1/r$. Otherwise, $\pi(\omega) \equiv 1$ and $\psi = \nu$. In any event, $\langle z_n \rangle$ is $\ell^1(\psi)$-bounded and converges almost surely to the zero sequence. Thus $\langle z_n \rangle$

has limit points in $\ell^1(\psi)^{**}$ consisting of pure charges. Letting φ denote any of these limit-point charges, we have $\varphi[\{\omega\}] = 0$ for every $\omega \in \mathbb{N}$ and $v = \varphi[\mathbb{N}] = 1$.[24]

5. Stochastic applications

In this section, we adjust the generic setup slightly in order to accommodate stochastic applications more naturally. First, we assume the $\mu = P$ is a probability measure. Second, we assume there is a payout β for which $E^P[\pi\,\beta] = 1$. Third, we use $\pi\,\beta$ as the Radon–Nikodym derivative to change to an equivalent probability measure Q. Thus, the value of a payout z_n is

$$v_n = \textstyle\int_\Omega \pi\, z_n \, d\mu = E^P[\pi\, z_n] = E^Q[x_n],$$

where $\frac{dQ}{dP} = \pi\,\beta$ and $x_n = z_n/\beta$. Fourth, we assume $E^Q[x_n] = x_0$ for all n.

Our arbitrage-related definitions carry over to this setting with the measure ψ replaced by Q. The central implication is this: No sequence $\langle x_n \rangle$ satisfying the four assumptions generates an arbitrage if $h(w) = x_0$. (Recall $h(w)$ is the cost of w.)

Given the four assumptions, if $\langle x_n \rangle$ is $L^1(Q)$-bounded and converges in Q-probability to $\widetilde{x}$, then $\langle x_n \rangle$ has weak* limit points in $L^1(Q)^{**}$, x_0 is the value of the limit points, $E^Q[\widetilde{x}]$ is the fundamental value, and $x_0 - E^Q[\widetilde{x}]$ is the bubble value. Note that if $x_n \geq 0$ for all n, then $\langle x_n \rangle$ is $L^1(Q)$-bounded.

The doubling strategy. We introduce the doubling strategy in a timeless setting. Assume the probability space $(\Omega, \mathcal{F}, P)$ contains a sequence of events $\langle e_n \rangle$ involving the flipping a coin, where

$$e_n \triangleq \text{the first head occurs on or before the } n\text{-th flip.}$$

Define the following random variable (payout):

$$x_n = \begin{cases} \alpha & \text{if } e_n \\ \alpha\,(1 - 2^n) & \text{if not } e_n, \end{cases}$$

where $\alpha > 0$ is an arbitrary constant.[25] The doubling strategy refers to the limiting payout as $n \to \infty$.

Rather than first specify the price system and the relevant probabilities under P and then explicitly change measure, we specify the probabilities directly under Q. In particular, we assume $Q[e_n] = 1 - (1/2)^n$, so that $E^Q[x_n] = 0$ for all n. Moreover, $E^Q[|x_n|] = 2\,\alpha\,(1 - (1/2)^n) \leq 2\,\alpha$, which establishes $L^1(Q)$-boundedness

[24] In the chest-of-drawers example, the fundamental component $\widetilde{z}$ is the zero sequence (all drawers closed), and φ is a function that ignores the open/closed status of drawers; it assigns zero weight to every finite collection of drawers, but assigns the value one to the infinite collection of all drawers. In other words, (the value assigned to) the whole is greater than the sum of (the values assigned to) the parts.

[25] One can imagine generating x_n sequentially as the coin is flipped: Make a bet that pays α if heads occurs on the first flip and pays $-\alpha$ if tails occurs instead. Stop if heads occurs (and keep α); otherwise borrow α to pay the loss and double the bet to $2\,\alpha$ for the next coin flip. Stop if heads occurs (pay off the debt and keep α); otherwise borrow $3\,\alpha$ to payoff the loss and the accumulated debt and redouble the bet to $4\,\alpha$. Continue this pattern, but stop after the n-th coin flip regardless of the outcome.

of $\langle x_n \rangle$. In addition, $\langle x_n \rangle$ converges in probability to α, but does not converge in norm. Therefore, the value (*i.e.,* marginal utility) of the doubling strategy is zero; the fundamental value is α and the bubble value is $-\alpha$. Thus the limit points are not in the positive cone of $(L^1)^{**}$; hence the doubling strategy does not present an arbitrage opportunity.

We can, of course, interpret the sequence of random variables $\langle x_n \rangle$ as a stochastic process (where time is indexed by n) with an initial value $x_0 = 0$. Define the process stopped at time τ as follows:

$$x_n^\tau \triangleq x_n \, 1_{\{n<\tau\}}(n) + x_\tau \, 1_{\{n\geq\tau\}}(n).$$

For $\tau \in \mathbb{N}$, the stopped process $\langle x_n^\tau \rangle$ is a martingale. Therefore, $\langle x_n \rangle$ is a local martingale, but not itself a martingale since it does not converge in norm.

Continuous-time finance. We adopt the assumptions in Chapter 6 of Duffie (1996), to which we refer the reader for omitted details. Take as given a complete probability space $(\Omega, \mathcal{F}, P)$, a standard d-dimensional Brownian motion B, a trading interval $\mathcal{T} = [0, T]$, a filtration $\mathbb{F} = \{\mathcal{F}_t : t \in \mathcal{T}\}$ satisfying the *usual conditions*[26] and for which $\mathcal{F}_T = \mathcal{F}$.

Consider the utility function

$$U(c) = E^P \left[\int_0^T u(c(t), t) \, dt + u^T(c_T, T) \right]$$

and its associated marginal utility:

$$\nabla U(c; z) = E^P \left[\int_0^T u_c(c(t), t) \, z(t) \, dt + u_c^T(c_T, T) \, z_T \right].$$

Suppose $u_c(c(0), 0) = 1$ and $u_c^T(c_T, T) = u_c(c(T), T)$. Let $\pi(t) = u_c(c(t), t)$ and assume $\pi \in L^2(P)$. Thus

$$\nabla U(c; z) = E^P \left[\int_0^T \pi(t) \, z(t) \, dt + \pi(T) \, z_T \right].$$

Assume there are $N \leq d+1$ (basis) securities whose values $V = (V_i)_{i=1}^N \in L^2(P)^N$ are Ito processes:

$$V(t) = V(0) + \int_0^t \mu_V(s) \, ds + \int_0^t \sigma_V(s) \, dW(s),$$

where σ_V is an $N \times d$ matrix. Assume the cumulative dividend processes for the basis securities is given by $D(t) = \int_0^t \delta(s) \, ds$. If π is a *state–price deflator*, then $G_\pi(t) \triangleq \pi(t) \, V(t) + \int_0^t \pi(s) \, \delta(s) \, ds$ is an $L^1(P)^N$ martingale and the basis securities are priced according to their values: $V(0) = E^P[G_\pi(T)]$.

Suppose the first security is strictly positive, pays no dividends, and has value one: $V_1(0) = 1 = E^P[\pi(T) \, V_1(T)]$. Let $Y(t) \triangleq 1/V_1(t)$ and define $G_Y(t) \triangleq Y(t) \, V(t) +$

[26] A filtration is said to satisfy the usual conditions if it is right-continuous and $\mathcal{F}_0$ contains the all the P-negligible events in $\mathcal{F}$. See Karatzas and Shreve (1991) for additional information.

$\int_0^t Y(s)\,\delta(s)\,ds$. By Girsanov's theorem, G_Y is an $L^1(Q)^N$-martingale under the equivalent probability measure Q, where $\frac{dQ}{dP} = \pi(T)/Y(T)$. Note $(G_Y)_1 \equiv 1$.

Let $M(t) \triangleq Y(t)\,\theta(t)^\top V(t)$ be the deflated value of a self-financing trading strategy $\theta = (\theta_1, \gamma)$, for which

$$M(t) = M(0) + \int_0^t \gamma(s)^\top dZ(s), \tag{5.1}$$

where $\gamma = (\theta_i)_{i=2}^N$, $Z = (G_{Y\,i})_{i=2}^N$, and $M(0) = \theta(0)^\top V(0)$ is the initial investment required by the trading strategy. The self-financing condition (5.1) is maintained by

$$\theta_1(t) = M(0) + \textstyle\int_0^t \gamma(s)^\top dZ(s) - Y(t) \sum_{i=2}^N \theta_i(t)\, V_i(t). \tag{5.2}$$

For γ in the class of processes for which the stochastic integral in (5.1) is defined, M is a local martingale. Therefore there is an increasing sequence of stopping times $\tau_n \geq 0$ such that $\lim_{t \to T}\ \tau_n = T$ a.s. that reduces M. In other words, $M_n(t) \triangleq M(t)\, 1_{\{t<\tau_n\}}(t) + M(\tau_n)\, 1_{\{t \geq \tau_n\}}(t)$ is a Q-martingale for all n. Note that $M_n(t) = Y(t)\,\theta_n(t)^\top V(t)$, where

$$\theta_{1\,n}(t) \triangleq \theta_1(t)\, 1_{\{t<\tau_n\}}(t) + M(\tau_n)\, 1_{\{t\geq\tau_n\}}(t) \quad \text{and} \quad \gamma_n(t) \triangleq \gamma(t)\, 1_{\{t<\tau_n\}}(t) \tag{5.3}$$

is a self-financing trading strategy. This trading strategy freezes the deflated gain at τ_n by investing all proceeds in the first asset, the deflated value of which is identically one by construction.

We can make the following three identifications: $x_0 = M(0)$ (the trading strategy's initial investment), $x_n = M_n(T)$ (the terminal value of each self-financing trading strategy in the sequence), and $\widetilde{x} = M(T)$ (the probability limit of the payoffs).[27] As such, we have a sequence $\langle x_n \rangle$ that converges in Q-probability to $\widetilde{x}$ and for which $E^Q[x_n] = x_0$ for all n. Therefore, there are no arbitrages. Moreover, if M is $L^1(Q)$-bounded, then $\langle M_n(T) \rangle$ has weak* limit points in $L^1(Q)^{**}$; the value of the limit points is $M(0)$; the value of the fundamental component $M(T)$ is $E^Q[M(T)]$; and the value of the bubble component is $M(0) - E^Q[M(T)]$.

Example: Nonnegative local martingales. Suppose M is a nonnegative local martingale. As such, it is $L^1(Q)$-bounded. Moreover, it is a supermartingale; thus $M(0) \geq E^Q[M(T)]$, with strict inequality if and only if the limiting payout contains a pure charge.

For example, consider the nonnegative local martingale presented by Loewenstein and Willard (2000):

$$M(t) = \exp\left\{ -\frac{1}{2} \int_0^t h(s)^2\, ds - \int_0^t h(s)\, d\widehat{W}(s) \right\},$$

where $h(t) = 1/(T-t)^{3/2}$ and $\widehat{W}$ is a scalar Brownian motion under Q. We can compute the distribution of $M(t)$ conditional on $\mathcal{F}_0$. In order to compute this

[27] For sufficiently integrable trading strategies, $\langle M_n(T) \rangle$ converges in L^2, which implies convergence in L^1 and in probability. For less integrable trading strategies, stochastic integration is *defined* in terms of convergence in probability.

distribution, first note that $\log(M(t))$ is normally distributed with mean $-\frac{1}{2}\,g(t)$ and variance $g(t)$, where

$$g(t) = \int_0^t h(s)^2\,ds = \frac{1}{2}\left(\frac{1}{(T-t)^2} - \frac{1}{T^2}\right).$$

Therefore, $M(t)$ is lognormally distributed with the following density:

$$f(x,t) = \frac{\exp\left\{\frac{-(g(t)+2\,\log(x))^2}{8\,g(t)}\right\}}{x\,\sqrt{2\,\pi\,g(t)}}. \tag{5.4}$$

For $0 < t < T$, the mean of the distribution is unity and the variance is $e^{g(t)} - 1$, which is unbounded as $t \to T$. The distribution converges in probability to zero as $t \to T$. The distribution of $M_n(T) = M(\tau_n)$ is $f(\,\cdot\,,\tau_n)$ as given by (5.4). Consequently, $E^Q[M_n(T)] = 1$ for all n, and therefore the fundamental value is zero and the bubble value is one.

The doubling strategy in continuous time. For completeness, we treat the doubling strategy in continuous time.[28]

Consider the process $y(t) \triangleq \int_0^t (T-u)^{-1/2}\,d\widehat{W}(u)$ for $0 \le t < T$, where $\widehat{W}$ is a standard univariate Brownian motion under Q. Note y is a martingale: $E^Q[y(t)] = y(0) = 0$; moreover, conditional on $\mathcal{F}_0$, $y(t)$ is normally distributed with mean zero and variance $\mathcal{V} = E^Q[y(t)^2] = -\log(1 - t/T)$. Let $0 < \alpha < \infty$ and consider the stopping time $\tau = \inf\{t : y(t) = \alpha\}$. Since y is a martingale, the stopped process $y^\tau(t) \triangleq y(t)\,1_{\{t<\tau\}}(t) + \alpha\,1_{\{t\ge\tau\}}(t)$ is a martingale as well; in particular, $E^Q[y^\tau(t)] = 0$ for $0 \le t < T$. We now show that y^τ converges in probability to α. Note $Q[|y^\tau(t) - \alpha| > 0] = 1 - Q[\tau < t]$. Adapting formula (6.2) of Karatzas and Shreve (1991, p. 80),

$$Q[\tau < t] = 2\,Q[y(t) > \alpha] = \sqrt{\frac{2}{\pi}}\int_{\alpha\,\mathcal{V}^{-1/2}}^{\infty} e^{-x^2/2}\,dx.$$

Thus $\lim_{t\uparrow T}\ Q[\tau < t] = 1$, and consequently $\lim_{t\uparrow T}\ Q[|y^\tau(t) - \alpha| > \varepsilon] = 0$ for all $\varepsilon > 0$, completing the demonstration. However, y^τ does not converge in the L^1 norm since $E^Q[|y^\tau(t) - \alpha|] = \alpha$ for all $t < T$.

The doubling strategy refers to the process $M(t) \triangleq y^\tau(t)$ for $0 \le t \le T$, where $M(T) = y^\tau(T)$. Clearly M is not a martingale (even though y^τ is) since $0 = M(0) \ne E^Q[M(T)] = \alpha$. Nevertheless, M is a local martingale. Define the sequence of stopping times $t_n \triangleq T\,(1 - (1/2)^n)$ and the stopped processes $M_n(t) \triangleq M(t)\,1_{\{t<t_n\}}(t) + M(t_n)\,1_{\{t\ge t_n\}}(t)$. To see that M is a local martingale, note that (i) $\lim_{n\to\infty}\ t_n = T$ and (ii) $E^Q[M_n(T)] = E^Q[M(t_n)] = E^Q[y^\tau(t_n)] = 0$

[28]See Duffie (1996, pp. 103–104) for a standard treatment.

for all n.[29] Note $\langle M_n \rangle$ is L^1-bounded, since $E^Q[|M_n(T)|] = 2\,E^Q[M(t_n)^+] \leq 2\,\alpha$ for all n.[30]

Therefore, $\langle M_n(T) \rangle$ has limit points in $(L^1)^{**}$, all of which have zero value, the limit of the expectations. The value of the fundamental component is the expectation of the probability limit, $\alpha = E^Q[M(T)]$, and the value of the bubble component is the difference, $-\alpha = 0 - \alpha$. Although the fundamental component is in L^1_+, the limit points are not in $(L^1)^{**}_+$. Consequently, the doubling strategy is not an arbitrage.

Appendix A. Theorems and proofs

Theorems A.1–A.4 collect established results regarding charges and measures. Theorem A.1 states that **ca** and **pch** are orthogonal complements. Theorem A.2 characterizes pure charges. Theorems A.3 and A.4 establish the links between the space of charges and the space of bounded linear functionals on L^∞ and between the space of signed measures and the space of integrable functions.

Theorem A.1 (Yosida–Hewitt)**.** *Any charge $\xi \in \mathbf{ba}$ can be expressed as $\xi = \rho + \varphi$, where $\rho \in \mathbf{ca}$ and $\varphi \in \mathbf{pch}$ are unique.*

Proof. See Bhaskara Rao and Bhaskara Rao (1983, 10.2.1, p. 241). □

Theorem A.2. *A bounded charge φ is a pure charge on $\mathcal{F}$ if and only if for every bounded measure ρ on $\mathcal{F}$ and $\varepsilon > 0$, there exists $A \in \mathcal{F}$ such that*

$$\varphi[A] = 0 \quad \textit{and} \quad \rho[\Omega \setminus A] < \varepsilon.$$

Proof. See Bhaskara Rao and Bhaskara Rao (1983, 10.3.3, p. 244). □

Theorem A.3. *There exists an isometric isomorphism between $L^1(\Omega, \mathcal{F}, \psi)^{**}$ and $\mathbf{ba}(\Omega, \mathcal{F}, \psi)$ determined by*

$$w \cdot f = \textstyle\int_\Omega f\,d\xi$$

where $w \in L^1(\Omega, \mathcal{F}, \psi)^{**}$, $f \in L^\infty(\Omega, \mathcal{F}, \psi)$, and $\xi \in \mathbf{ba}(\Omega, \mathcal{F}, \psi)$.

Proof. See Dunford and Schwartz (1958, IV.8.16, p. 296). □

Theorem A.4 (Radon–Nikodym)**.** *The identity*

$$\rho[A] = \textstyle\int_A \widetilde{z}\,d\psi \quad \textit{for } A \in \mathcal{F} \tag{A.1}$$

determines an isometric isomorphism between $\mathbf{ca}(\Omega, \mathcal{F}, \psi)$ and $L^1(\Omega, \mathcal{F}, \psi)$ where $\widetilde{z} \in L^1$ corresponds to $\rho \in \mathbf{ca}$.

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[29] In a complete market, there is a self-financing trading strategy that replicates $M(t)$ for $t \in [0, T]$. For example, consider the Black–Scholes world where V is the value of the stock. For simplicity, assume $\beta = \pi = 1$, which implies $P = Q$. Additionally assume $dV = V\,d\widehat{W}$ with $V(0) = 1$. The self-financing trading strategy (for which $\theta_1 + \theta_2\,V = M$) is given by $\theta_1(t) = \int_0^t \theta_2(u)\,dV(u) - \theta_2(t)\,V(t)$ as per (5.2) and $\theta_2(t) = \big(V(t)\,\sqrt{T-t}\big)^{-1}\,1_{\{t<\tau\}}(t)$. The sequence of the self-financing trading strategies that replicate $\langle M_n \rangle$ is given by (5.3).

[30] Define $f^+ \triangleq \max[f, 0]$ and $f^- \triangleq -\min[f, 0]$. Clearly, $f^+, f^- \geq 0$. Note $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Therefore, if $\langle \int f_n^+\,d\mu \rangle$ and $\langle \int f_n\,d\mu \rangle$ are bounded, then $\langle f_n \rangle$ is L^1-bounded. Moreover, if $f_n = f_n^+$ or $f_n = f_n^-$ and if $\langle \int f_n\,d\mu \rangle$ is bounded, $\langle f_n \rangle$ is L^1-bounded.

Proof. See Dunford and Schwartz (1958, III.10.2, p. 176). [Note: The payout $\widetilde{z}$ in (A.1) is the Radon–Nikodym derivative of ρ with respect to ψ: $\frac{d\rho}{d\psi} = \widetilde{z}$.] □

Theorem A.5 characterizes the relation between almost sure convergence and convergence in measure on the one hand and the fundamental component on the other. As far as we know, Theorem A.5 is new.

Theorem A.5. *Let $(\Omega, \mathcal{F}, \psi)$ be a positive, finite measure space, $\langle z_\alpha \rangle \subset L^1$ a net weak* converging to $w \in (L^1)^{**}$, $\xi \in$ **ba** corresponding to w, and $\xi = \rho + \varphi$, with $\rho \in$ **ca** and $\varphi \in$ **pch** (the Yosida-Hewitt decomposition of ξ). Let $\widetilde{z}$ denote the element in L^1 corresponding to ρ. Then $\widetilde{z}$ is both (a) the limit of $\langle z_\alpha \rangle$ in the topology of convergence in measure, and (b) its only limit point in the topology of almost-sure convergence.*

Proof. Note first that $\langle z_\alpha \rangle$ converges to ξ if and only if $\langle z_\alpha - \widetilde{z} \rangle$ converges to φ, so that we can suppose that $\rho = 0$ (and thus $\widetilde{z} = 0$). Second, $\langle z_\alpha \rangle$ converges to φ if and only if $\langle z_\alpha^+ \rangle$ converges to φ^+ and $\langle z_\alpha^- \rangle$ converges to φ^-, so that it is sufficient to consider the case where $\langle z_\alpha \rangle$ is a positive net converging to a pure charge.

Because φ is a pure charge, ψ is finite, and φ vanishes on sets of ψ-measure zero, there exists a sequence $\langle A_n \rangle \uparrow \Omega$ where $A_n \in \mathcal{F}$ such that $\varphi[A_n] = 0$ for all n (Bhaskara Rao and Bhaskara Rao 1983, Theorem 10.3.3).

By definition, $\langle z_\alpha \rangle$ weak* converges to φ if and only if $\langle \int_\Omega f\, z_\alpha \, d\psi \rangle$ converges to $\int_\Omega f \, d\varphi$ for all $f \in L^\infty$. In particular, using the test function 1_{A_n}, $\langle \int_{A_n} z_\alpha \, d\psi \rangle$ converges to $\int_{A_n} d\varphi = \varphi[A_n] = 0$. Therefore, for any $\varepsilon > 0$ there exists α such that $\beta > \alpha$ implies $\psi[\{\omega \in A_n : z_\beta(\omega) > \varepsilon\}] < \varepsilon/2$. Choosing n such that $\psi[\Omega \backslash A_n] < \varepsilon/2$, we conclude that, for sufficiently large β,

$$\psi[\{\omega \in \Omega : z_\beta(\omega) > \varepsilon\}] < \varepsilon/2 + \psi[\{\omega \in A_n : z_\beta(\omega) > \varepsilon\}] < \varepsilon.$$

In other words, $\langle z_\alpha \rangle$ converges in ψ-measure to 0, which proves (a). To get (b), note that there is a subnet converging to $\widetilde{z}$ almost surely (see, *e.g.,* Royden (1988, p. 95).), so that $\widetilde{z}$ is a limit point of $\langle z_\alpha \rangle$. Since almost-sure convergence implies convergence in measure and all subnets converge in measure to $\widetilde{z}$, there cannot be any other almost-sure limit point. □

Theorems A.6–A.10 deal with the convergence of sequences $\langle z_n \rangle$ of payouts. These theorems are either well-known or trivial.

Theorem A.6. *If $\langle z_n \rangle$ is a sequence in $L^1(\psi)$ with $z_n \geq 0$, then $\langle z_n \rangle$ is L^1-bounded if and only if $\langle \mathcal{V}[z_n] \rangle$ is bounded.*

Proof. Since $z_n \geq 0$, $\|z_n\|_1^\psi = \mathcal{V}[z_n]$. □

Theorem A.7. *If a sequence is bounded in L^1, then it has weak* limit points in $(L^1)^{**}$.*

Proof. The unit ball of L^1 is contained in the unit ball of $(L^1)^{**}$. By Alaoglu's theorem, the unit ball of $(L^1)^{**}$ is weak* compact. □

Theorem A.8. *Given an $L^1(\psi)$-bounded sequence $\langle z_n \rangle$, if $v = \lim_{n\to\infty} \mathcal{V}[z_n]$ exists, then the value of the limit points is v.*

Proof. This follows from the definition of weak* convergence. □

Theorem A.9. *Given an $L^1(\psi)$-bounded sequence, if the sequence converges in measure to $\widetilde{z}$, then the fundamental component of the limit points is $\widetilde{z} \in L^1(\psi)$ and the fundamental value of the limit points is $\mathcal{V}[\widetilde{z}]$.*

Proof. This is a consequence of Theorem A.5. □

Theorem A.10. *If a sequence in L^1 converges in the norm topology, then the bubble component of the limit point is zero.*

Proof. A convergent sequence in L^1 is a convergent net in $(L^1)^{**}$. □

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