THE ESSENCE OF ASSET PRICING

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1. Introduction

The Nobel Prize for economics was shared this year by Robert Merton and Myron Scholes for their contributions to option pricing theory.[1] Had Fischer Black lived, he too would have shared the prize for the work cited by the prize committee. Since the early 1970s when Black and Scholes published their now famous option pricing model[2] and Merton published his extensions of their model,[3] there has been a remarkable unification of a variety of strands of asset-pricing theory. This note tries to provide some insight into the theory of asset-pricing that their seminal work inspired and the unified theory that has evolved.

Black and Scholes applied the notion that "you can't get something for nothing" (which is known more formally as "the absence of arbitrage opportunities") to solve the problem of finding the rational option price. Earlier, Franco Modigliani and Merton Miller (each of whom has won the Nobel prize on different occasions) had used arguments based on the absence of arbitrage opportunities to show that, absent taxes and transaction costs, the value of a firm is independent of its capital structure (*i.e.*, whether the firms finances its operations with debt or equity is irrelevant). Of course there *are* taxes and transaction costs, but the Modigliani–Miller propositions help us to focus on what's going on by clearly showing us what's *not* going on.

Early attempts to derive a comprehensive theory of asset pricing typically involved modeling investors' attitudes toward risk and return. Black and Scholes initiated the investigation of the consequences of conditions that ensure there are no arbitrage opportunities. Although this approach seems quite superficial by comparison, an intimate relationship between the two approaches was later discovered.

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The views expressed herein are the author's and do not necessarily reflect those of the Board of Governors of the Federal Reserve System.
[1]This sentence is a bit out of date.
[2]Black and Scholes (1973).
[3]Merton (1973).

In fact much can be learned about preference-based models from preference-free absence-of-arbitrage conditions.[4]

I will describe asset pricing in a simple setting. There are only two points in time to consider: right now and one year from now (the future). In the future there are only two possible states of the world. The simple binomial structure of the model was first introduced by William Sharpe (also a Nobel prize winner) in the first edition of his undergraduate-level textbook *Investments*.

In this simple setting, we assume that there are no taxes or transactions costs. These assumptions are in some sense unrealistic of course, but if we cannot understand how the world works without frictions it is doubtful that we can understand how it works when they are present.

2. A stock, a bond, and no arbitrage

A stock and a bond. We start with two financial assets: a risky stock and a risk-free bond. The value of a risk-free bond right now is B_{now}, and in one year the value will surely be $(1 + r)\, B_{\text{now}}$, where r is the risk-free interest rate. The value of the stock right now is S_{now}. In one year, there will be two possible *states of the world*: the value of the stock will either be S_{up} or S_{down}, where

$$0 < S_{\text{down}} < S_{\text{now}} < S_{\text{up}} < \infty.$$

Let the probabilities that the stock price goes up or down be given by $p_{\text{up}} > 0$ and $p_{\text{down}} > 0$, where $p_{\text{up}} + p_{\text{down}} = 1$. For now, we will not be any more specific about the values of the probabilities.

Portfolios. Consider buying the stock on margin: *i.e.*, borrowing money to buy stock. In our setting, borrowing money means selling bonds. We form a *portfolio* by buying (right now) x shares of the stock simultaneously selling y bonds. The cost of our portfolio is the net outlay right now, the cost of the shares of stock less the total amount borrowed,

$$\Pi_{\text{now}} = x\, S_{\text{now}} - y\, B_{\text{now}}.$$

How much will our portfolio be worth in one year? We will sell the stock and receive either $x\, S_{\text{down}}$ or $x\, S_{\text{up}}$, but we will have to repay our loan with interest, $y\,(1+r)\, B_{\text{now}}$. Thus, depending on whether the stock goes down or up, our portfolio will be worth either

$$\Pi_{\text{down}} = x\, S_{\text{down}} - y\,(1 + r)\, B_{\text{now}}$$

or

$$\Pi_{\text{up}} = x\, S_{\text{up}} - y\,(1 + r)\, B_{\text{now}}.$$

[4] In a later version, I will expand on early work done by Samuelson (*i.e.*, before Black, Scholes, and Merton) and general equilibrium asset pricing by Lucas (both Nobel prize winners).

Arbitrage opportunities with the stock and the bond. An *arbitrage* is a trading strategy that produces something for nothing. In our simple setting, a trading strategy is nothing more than a portfolio of the stock and the bond where we choose the portfolio weights once and for all, but in a more realistic setting it could involve changing the portfolio weights over time in response to movements in the stock price. Suppose for example we could form a portfolio that cost nothing today and guaranteed a positive payoff in all future states of the world. That would be an arbitrage. (In fact, even if it only had positive payoffs in some states it would still be an arbitrage—as long as it didn't have any negative payouts. This would be like free insurance against the occurrence of those states of the world.) An alternative arbitrage would be a trading strategy that generated something right now but required no payouts under any circumstances next year.

Consider the first sort of trading strategy in more detail: Buy one share of stock and borrow money to finance the entire cost, putting no money down. In this case, $x = 1$ and $y = S_{\text{now}}/B_{\text{now}}$, which makes the outlay today zero:

$$\Pi_{\text{now}} = x\,S_{\text{now}} - y\,B_{\text{now}} = S_{\text{now}} - \left(\frac{S_{\text{now}}}{B_{\text{now}}}\right) B_{\text{now}} = 0.$$

Now consider the payoffs. Given our portfolio weights, the trading strategy will payoff either

$$\Pi_{\text{down}} = S_{\text{down}} - (1 + r)\,S_{\text{now}} \tag{2.1a}$$

or

$$\Pi_{\text{up}} = S_{\text{up}} - (1 + r)\,S_{\text{now}}. \tag{2.1b}$$

If both payoffs are positive (or if $\Pi_{\text{up}} > 0$ and $\Pi_{\text{down}} \geq 0$), then we have an arbitrage—something positive later (a positive payoff in at least one state of the world next year with no negative payoffs) for nothing today. On the other hand, if both payoffs are negative (or if $\Pi_{\text{up}} \leq 0$ and $\Pi_{\text{down}} < 0$), then there is a different arbitrage—simply change the sign of our portfolio weights, $x = -1$ and $y = -S_{\text{now}}/B_{\text{now}}$. (These portfolio weights correspond to selling the stock short and lending the proceeds.)

Therefore, in order to ensure the absence of arbitrage opportunities involving the stock and the bond, we require that $\Pi_{\text{down}} < 0$ and $\Pi_{\text{up}} > 0$, so that one would face the possibility of losing money no matter which trading strategy one adopted. Given (2.1), we can write these conditions as

$$S_{\text{down}} < (1 + r)\,S_{\text{now}} < S_{\text{up}}. \tag{2.2}$$

In other words, the certain rate of return on the bond lies between the low and the high rates of return for the stock:

$$\frac{S_{\text{down}}}{S_{\text{now}}} - 1 < r < \frac{S_{\text{up}}}{S_{\text{now}}} - 1.$$

These conditions guarantee that neither the stock nor the bond *dominates* the other security (in the sense that the return on one security is greater than the other in all states).

3. Puts, calls, and forwards

In this section we will examine call options, put options, forward contracts, and put–call parity.[5]

Call option. The payoffs for—and therefore the value of—a derivative security are determined by (*i.e.*, derived from) the value of something else, called the *underlier*. The classic example of a derivative security is a *call option*. A call option is a contract between the option *holder* and the option *writer* that gives the holder the right to purchase a pre-specified number of shares of the underlier for a pre-specified amount (the *strike price*) on a pre-specified date (the *expiration date*) from the option writer. In other words, the holder has the right to *call* the underlier from the writer. (What I have described is a *European* call option. An *American* call option gives the holder the right of *early exercise*, the right to exercise anytime before the expiration date as well.) If the value of the underlier is less than the strike price on the expiration date, we say that the option finished *out of the money*. Because the option holder is not required to exercise the option, the holder will allow the option to expire unexercised in this case and the payoff will be zero. On the other hand, if the option finishes *in the money*, the holder will exercise it, pay the strike price, and receive the underlier which is worth more. Since the holder can now sell the underlier for its market value, the option's payoff in this case is the difference between the value of the underlier and the strike price.

We will assume that a call option is written on one share of stock. If the strike price for the option is K_C, then the payoffs for the call option in general are determined by

$$\max(S_{\text{expiration}} - K_C, 0),$$

where the function $\max(x, y)$ returns the maximum of x and y.[6]

We will examine the value of a call option on the stock in our simple setting. We want to find the price of the call option right now, C_{now}. First we need to determine the payoffs for the call:

$$\begin{aligned} C_{\text{down}} &= \max(S_{\text{down}} - K_C, 0) \\ C_{\text{up}} &= \max(S_{\text{up}} - K_C, 0). \end{aligned}$$

If $K_C \geq S_{\text{up}}$, then $C_{\text{up}} = C_{\text{down}} = 0$, which is not very interesting. To keep things interesting let

$$S_{\text{down}} < K_C < S_{\text{up}}. \tag{3.1}$$

For example, we could have an *at-the-money call* where $K_C = S_{\text{now}}$. The payoffs for the option in one year are easy to compute given the stock price one year from now. If the stock price goes down the option is worth nothing, but if the stock price

[5]Also talk about valuing corporate debt by treating equity as a call option, FDIC insurance by treating the bank as having a put option, and portfolio insurance.

[6]The function $\max(x, 0)$ is often written as $(x)^+$.

goes up the option is worth $S_{\text{up}} - K_C$:

$$\begin{aligned} C_{\text{down}} &= \max(S_{\text{down}} - K_C, 0) = 0 \\ C_{\text{up}} &= \max(S_{\text{up}} - K_C, 0) = S_{\text{up}} - K_C > 0. \end{aligned}$$

Delta hedging. The *delta* of an option measures how much the value of the option changes per unit of change in the stock price. In our setting, this amounts to

$$\Delta_C = \frac{C_{\text{up}} - C_{\text{down}}}{S_{\text{up}} - S_{\text{down}}} = \frac{S_{\text{up}} - K_C}{S_{\text{up}} - S_{\text{down}}}.$$

Delta hedging tells us how to form a portfolio of the stock and the call option to eliminate the risk of the portfolio. If we buy Δ_C shares of the stock and sell one call, the combined payoffs will be the same in each state:

$$\begin{aligned} S_{\text{up}}\,\Delta_C - C_{\text{up}} &= S_{\text{down}}\,\Delta_C \\ S_{\text{down}}\,\Delta_C - C_{\text{down}} &= S_{\text{down}}\,\Delta_C. \end{aligned}$$

Thus we have formed a risk-free portfolio, which cost $S_{\text{now}}\,\Delta_C - C_{\text{now}}$. The absence of arbitrage requires that this portfolio earn the risk-free rate. In other words we must have

$$(1 + r)\,(S_{\text{now}}\,\Delta_C - C_{\text{now}}) = S_{\text{down}}\,\Delta_C.$$

We solve this equation for the arbitrage-free price of the option:

$$\begin{aligned} C_{\text{now}} &= \frac{\big((1 + r)\,S_{\text{now}} - S_{\text{down}}\big)\,\Delta_C}{1 + r} \\ &= \frac{\big((1 + r)\,S_{\text{now}} - S_{\text{down}}\big)\,(S_{\text{up}} - K_C)}{(S_{\text{up}} - S_{\text{down}})\,(1 + r)}. \end{aligned}$$

Note that since we don't know what the probabilities of the up and down states are, we don't know what the expected return on the stock is. Even so, we have been able to figure out the price of the call.

Put option. A *put option* is similar to a call option except that is confers to the holder the right to *sell* the underlier to the writer at the strike price (to put it to him). Let the strike price for the put option, K_P also be between S_{down} and S_{up}. For the put, the payoffs are given by

$$\begin{aligned} P_{\text{down}} &= \max(K_P - S_{\text{down}}, 0) = K_P - S_{\text{down}} > 0 \\ P_{\text{up}} &= \max(K_P - S_{\text{up}}, 0) = 0. \end{aligned}$$

We can use delta hedging to solve for the put price right now, P_{now}. In this case

$$\Delta_P = \frac{P_{\text{up}} - P_{\text{down}}}{S_{\text{up}} - S_{\text{down}}} = \frac{-(K_P - S_{\text{down}})}{S_{\text{up}} - S_{\text{down}}}.$$

Notice that the delta for the put is negative. If we buy one put and sell Δ_P shares of the stock (since Δ_P is negative, this means buying shares of the stock), the combined payoffs will be the same in each state:

$$P_{\text{up}} - S_{\text{up}}\,\Delta_P = -S_{\text{up}}\,\Delta_P$$
$$P_{\text{down}} - S_{\text{down}}\,\Delta_P = -S_{\text{up}}\,\Delta_P.$$

Again, the absence of arbitrage requires that risk-free portfolios earn the risk-free interest rate,

$$(1+r)\,(P_{\text{now}} - S_{\text{now}}\,\Delta_P) = -S_{\text{up}}\,\Delta_P,$$

which delivers the value of the put right now:

$$P_{\text{now}} = \frac{(S_{\text{up}} - (1+r)\,S_{\text{now}})\,(K_P - S_{\text{down}})}{(S_{\text{up}} - S_{\text{down}})\,(1+r)}.$$

Forward contract. A forward contract is an agreement through which the *short* agrees to deliver the underlier to the *long* in exchange for the *forward price* (K_F) on the *delivery date*. On the delivery date, the long pays K_F and receives the underlier, which is worth either S_{up} or S_{down}. Thus, the payoffs for the long are

$$F_{\text{down}} = S_{\text{down}} - K_F < 0$$
$$F_{\text{up}} = S_{\text{up}} - K_F > 0.$$

Since no money changes hands at the inception of a forward contract, its value at inception must be zero. In fact, the forward price is set to make the value of a forward contract zero. Clearly, a forward contract is not an option, since some of its payoffs are negative. Nevertheless, we will see how to construct a forward contract out of options. The delta for the forward contract is

$$\Delta_F = \frac{(S_{\text{up}} - K_F) - (S_{\text{down}} - K_F)}{S_{\text{up}} - S_{\text{down}}} = 1.$$

If we buy one share of stock and sell one forward contract, then the payoffs for the portfolio are

$$S_{\text{up}} - F_{\text{up}} = K_F$$
$$S_{\text{down}} - F_{\text{down}} = K_F,$$

and therefore the return on the portfolio must equal the risk-free rate:

$$(1+r)\,(S_{\text{now}} - F_{\text{now}}) = K_F.$$

Now here's the twist: Instead of taking the forward price (K_F) as given and solving for F_{now}, we set $F_{\text{now}} = 0$ and solve for K_F:

$$K_F = (1+r)\,S_{\text{now}}. \tag{3.2}$$

Equation (3.2) gives the forward price. Recalling the absence of arbitrage condition for the stock and the bond (2.2), we see that this condition amounts to requiring that the forward price for the stock lie between S_{down} and S_{up} (*i.e.*, within its support).

Put–call parity and synthetic forwards. There are definite relationships among these three derivative securities. We can *synthesize* a forward contract by (i) buying a call option with strike price $K_C = K$, (ii) selling a put option with the same strike price $K_P = K$, and (iii) borrowing the money to pay the net cost right now $C_{\text{now}} - P_{\text{now}}$. The payoffs to this portfolio look like the payoffs to a forward contract:

$$\begin{aligned}\Pi_{\text{down}} &= C_{\text{down}} - P_{\text{down}} - \big\{(1+r)\,(C_{\text{now}} - P_{\text{now}})\big\}\\ &= S_{\text{down}} - K_F\end{aligned}$$

and

$$\begin{aligned}\Pi_{\text{up}} &= C_{\text{up}} - P_{\text{up}} - \big\{(1+r)\,(C_{\text{now}} - P_{\text{now}})\big\}\\ &= S_{\text{up}} - K_F,\end{aligned}$$

where the synthetic forward price K_F equals the common strike price on the options plus the loan repayment:

$$K_F = K + (1+r)\,(C_{\text{now}} - P_{\text{now}}). \tag{3.3}$$

Now recall from (3.2) that $K_F = (1+r)\,S_{\text{now}}$. We can substitute this expression into (3.3) and solve for the call price:

$$C_{\text{now}} = P_{\text{now}} + S_{\text{now}} - \frac{K}{1+r}. \tag{3.4}$$

Equation (3.4) is known as *put–call parity.*

4. Derivative securities and no arbitrage

Now consider any security whose payoffs one year from now depend only on whether the stock price went up or down. We call such a security a derivative because its value is derived from the value of the underlier (the stock, in this case). Suppose the payoffs of the derivative in one year are D_{down} if the stock price goes down and D_{up} is the stock price goes up. For now we will consider a general derivative security whose payoffs are not explicitly linked to the underlying stock.[7]

The central point of pricing derivatives is this. If we can replicate the payoffs of the derivative using a portfolio of the stock and the bond so that $D_{\text{up}} = \Pi_{\text{up}}$ and $D_{\text{down}} = \Pi_{\text{down}}$, then the price of the derivative right now must equal the price of the replicating portfolio right now:

$$D_{\text{now}} = \Pi_{\text{now}} = x\,S_{\text{now}} - y\,B_{\text{now}}. \tag{4.1}$$

Otherwise there would be arbitrage opportunities involving the stock, the bond, and the derivative security. For example, if $D_{\text{now}} < \Pi_{\text{now}}$ you could buy the derivative, "sell" the replicating portfolio, and pocket the difference. Because the portfolio replicates the payoffs of the derivative, you know for sure that you will not have any net cash flows next year. Therefore you got something today for nothing later.

Now let's see how to replicate the derivative's payoffs. We must choose x and y in order to guarantee that the portfolio identically replicates the payoffs of the derivative security in one year. (Of course, the values of x and y will be different

[7] We are pursuing non-Markovian representations here.

for different derivative securities.) In order to replicate the payoffs of the derivative, the portfolio weights, x and y, must satisfy

$$D_{\text{down}} = x\, S_{\text{down}} - y\,(1+r)\, B_{\text{now}} \tag{4.2a}$$

and

$$D_{\text{up}} = x\, S_{\text{up}} - y\,(1+r)\, B_{\text{now}}. \tag{4.2b}$$

Equations (4.2) comprise two equations in the two unknown portfolio weights, x and y. The solution to (4.2) is

$$\begin{aligned} x &= \frac{D_{\text{up}} - D_{\text{down}}}{S_{\text{up}} - S_{\text{down}}} \\ y &= \frac{D_{\text{up}}\, S_{\text{down}} - D_{\text{down}}\, S_{\text{up}}}{(S_{\text{up}} - S_{\text{down}})\,(1+r)\, B_{\text{now}}}. \end{aligned} \tag{4.3}$$

With these values for x and y, the value of the derivative security right now must be the value of the portfolio right now. In other words, insert the values for x and y in (4.3) into the expression for D_{now} in (4.1):

$$\begin{aligned} D_{\text{now}} &= x\, S_{\text{now}} - y\, B_{\text{now}} \\ &= \left(\frac{D_{\text{up}} - D_{\text{down}}}{S_{\text{up}} - S_{\text{down}}}\right) S_{\text{now}} - \left(\frac{D_{\text{up}}\, S_{\text{down}} - D_{\text{down}}\, S_{\text{up}}}{(S_{\text{up}} - S_{\text{down}})\,(1+r)\, B_{\text{now}}}\right) B_{\text{now}} \\ &= \frac{\hat{p}_{\text{up}}\, D_{\text{up}} + \hat{p}_{\text{down}}\, D_{\text{down}}}{1+r}, \end{aligned} \tag{4.4a}$$

where

$$\begin{aligned} \hat{p}_{\text{up}} &= \frac{(1+r)\, S_{\text{now}} - S_{\text{down}}}{S_{\text{up}} - S_{\text{down}}} \\ \hat{p}_{\text{down}} &= 1 - \hat{p}_{\text{up}} \end{aligned} \tag{4.4b}$$

Before we try to interpret the solution (4.4) in detail, first consider the value of a very simple security that is closely related to the bond we have already modeled. Consider a *zero-coupon bond.* A zero-coupon bond pays one unit with certainty when it matures, which in our example is one year from now. Since for a zero-coupon bond $D_{\text{up}} = D_{\text{down}} = 1$, our formula tells us that $D_{\text{now}} = 1/(1+r)$. This is known as the *present value* of one dollar to be paid one year from now with certainty.

Now let's turn to interpreting the solution given in (4.4). Now notice that condition (2.2) guarantees that $\hat{p}_{\text{up}} > 0$ and $\hat{p}_{\text{down}} > 0$. Also notice that $\hat{p}_{\text{up}} + \hat{p}_{\text{down}} = 1$. These properties allow us to treat $\hat{p}_{\text{up}}$ and $\hat{p}_{\text{down}}$ as if they were *probabilities.* Given this treatment, we can interpret the value of the derivative right now, D_{now}, as the *present value* of its *average payoff*, where we are using $\hat{p}_{\text{up}}$ and $\hat{p}_{\text{down}}$ as the probabilities in the average. But note that these probabilities are not the actual probabilities that the stock goes up or down in value in the real world. These are *pseudo* probabilities that have been constructed purely from the dynamics of asset prices. (We will examine the relationship between these pseudo probabilities and the real-world probabilities below.) Now we can write the solution as

$$D_{\text{now}} = \frac{\hat{D}_{\text{avg}}}{1+r}, \tag{4.5a}$$

where

$$\hat{D}_{\text{avg}} = \hat{p}_{\text{up}}\, D_{\text{up}} + \hat{p}_{\text{down}}\, D_{\text{down}}. \tag{4.5b}$$

We will call equation (4.5) the *pricing formula*.

The pricing formula applies to *all* assets, including the stock itself. In this case, of course, $D_{\text{up}} = S_{\text{up}}$ and $D_{\text{down}} = S_{\text{down}}$. And, of course, the pricing formula delivers the correct answer: $D_{\text{now}} = S_{\text{now}}$.

Let's rearrange the pricing formula as follows:

$$\hat{D}_{\text{avg}} - D_{\text{now}} = r\, D_{\text{now}}. \tag{4.6}$$

Equation (4.6) says that the expected change in the value of the asset—calculated using the pseudo probabilities—equals the current value of the asset times the risk-free interest rate. If D_{now} is positive (it need not be) then we can write

$$\frac{\hat{D}_{\text{avg}}}{D_{\text{now}}} - 1 = r. \tag{4.7}$$

Equation (4.7) shows that the average return for *any* security with a positive value right now—calculated using the pseudo probabilities—is the risk-free interest rate, regardless of the perceived risk of the security. Hence these pseudo probabilities are often called *risk-neutral* probabilities, and equation (4.5) is often referred to as risk-neutral pricing.

There is another way we can think about our pricing formula. Divide the price of the derivative security right now by the price of the bond right now and divide the payoffs of the derivative security next year by the payoff of the bond next year. Let

$$Z_{\text{now}} = \frac{D_{\text{now}}}{B_{\text{now}}}, \quad Z_{\text{up}} = \frac{D_{\text{up}}}{(1+r)\, B_{\text{now}}}, \quad \text{and} \quad Z_{\text{down}} = \frac{D_{\text{down}}}{(1+r)\, B_{\text{now}}}.$$

We refer to these Z values as the *deflated* values of the derivative security. Using these deflated values, we can rewrite our pricing formula in the following extremely compact way:

$$Z_{\text{now}} = \hat{Z}_{\text{avg}}. \tag{4.8}$$

Pricing formula (4.8) says that the value of deflated security right now equals the average value of deflated security next year. Anything with the property that its future average value equals its current value is called a *martingale*, and so formula (4.8) is known as the martingale pricing formula. Of course, formulas (4.4), (4.5), and (4.8) are all equivalent: They simply say the same thing in different ways.[8]

[8]It turns out that the solution to many problems in physics can be expressed exactly as formula (4.8). Also Feynman–Kac in a Markovian setting.

Arrow–Debreu securities and state prices. Now consider two fundamental securities from which all other securities can be easily constructed. These are known as *Arrow–Debreu securities.* Kenneth Arrow and Gerard Debreu each independently thought up these securities. (And each won the Nobel prize in Economics independently.) The first security pays one unit if the stock goes up and nothing if the stock goes down, $D_{\text{up}} = 1$ and $D_{\text{down}} = 0$, while the second one pays nothing if the stock goes up and one unit if the stock down, $D_{\text{up}} = 0$ and $D_{\text{down}} = 1$. The prices of these securities are called *state prices* since they measure the price of a unit payoff in each state. From (4.5) we can see that the two state prices are

$$\frac{\hat{p}_{\text{up}}}{1+r} \quad \text{and} \quad \frac{\hat{p}_{\text{down}}}{1+r}. \tag{4.9}$$

As mentioned above, the Arrow–Debreu securities are the building blocks out of which all securities can be constructed. Since the stock's payoffs are S_{up} and S_{down}, the value of the stock right now must be S_{up} times the up-state price plus S_{down} times the down-state price. Note that if all state prices are positive, one can never get something for nothing. In fact, our absence-of-arbitrage condition (2.2) is nothing more than the condition required to guarantee all the state prices are positive.

Complete markets. We have been working in an economy with a *complete market.* What this means is that we have enough securities (with independent returns) to form portfolios that can pinpoint every Arrow–Debreu security. If instead, we had only the stock and no bond, then we would have an *incomplete market*, and we would be unable to uncover the state prices using the securities available. Pure absence-of-arbitrage could not tell us how to price all claims that might occur. Nevertheless, we can still apply general equilibrium asset pricing, which we discuss in detail below.

Extracting the pseudo probabilities with option prices. We can use put prices to extract the pseudo probabilities. Consider two put options with different strike prices, K_P^1 and K_P^2, where both strike prices are between the two possible stock prices: $S_{\text{down}} < K_P^i < S_{\text{up}}$. The price of these puts is given by

$$P_{\text{now}}^i = \frac{\hat{p}_{\text{down}}\,(K_P^i - S_{\text{down}})}{1+r}.$$

We can use these two put prices to extract the probability:

$$(1+r)\left(\frac{P_{\text{now}}^2 - P_{\text{now}}^1}{K_P^2 - K_P^1}\right) = \hat{p}_{\text{down}}.$$

If we apply this formula where $K_P^i < S_{\text{down}}$ we get 0, and if we apply it where $K_P^i > S_{\text{down}}$ we get 1. The reader with a background in probability will recognize that we can trace out the cumulative distribution function (CDF) for the pseudo probabilities this way. Moreover, if we had a continuous state space with a differentiable CDF, then we could recover the pseudo probability density function with $\partial^2 P/\partial K^2$ (or $-\partial^2 C/\partial K^2$). There is a small industry today actively involved in doing just that.

Risk, return, and the price of risk. We have been able to solve for asset prices without referring explicitly to risk or return. Yet in an important sense the relation between risk and return is central asset pricing. In this section we will make explicit the implicit relation embedded in the conditions for no arbitrage.

First, we must define what we mean by risk and return. Using the true probabilities, which we now assume are $(\frac{1}{2}, \frac{1}{2})$, the average value of a asset's payoffs next year is

$$D_{\text{avg}} = p_{\text{up}}\, D_{\text{up}} + p_{\text{down}}\, D_{\text{down}} = \frac{D_{\text{up}} + D_{\text{down}}}{2}.$$

One measure of the risk of the asset is the *volatility* of its payoffs:

$$D_{\text{vol}} = (D_{\text{up}} - D_{\text{down}})\sqrt{p_{\text{up}}\, p_{\text{down}}} = \frac{D_{\text{up}} - D_{\text{down}}}{2}.$$

The *variance* of the return in its payoffs is given by

$$\begin{aligned} D_{\text{var}} &= p_{\text{up}}\,(D_{\text{up}} - D_{\text{avg}})^2 + p_{\text{down}}\,(D_{\text{down}} - D_{\text{avg}})^2 \\ &= \frac{(D_{\text{up}} - D_{\text{avg}})^2 + (D_{\text{down}} - D_{\text{avg}})^2}{2}. \end{aligned}$$

The *standard deviation* of its payoffs is given by

$$D_{\text{dev}} = \sqrt{D_{\text{var}}} = |D_{\text{vol}}|,$$

where $|x|$ denotes the absolute value of x. For those assets whose value is positive, $D_{\text{now}} > 0$, we can write (4.11) in terms of *returns*. Define the returns on the asset as follows:

$$d_{\text{up}} = \frac{D_{\text{up}}}{D_{\text{now}}} - 1 \quad \text{and} \quad d_{\text{down}} = \frac{D_{\text{down}}}{D_{\text{now}}} - 1.$$

Then it is easy to show that

$$d_{\text{avg}} = \frac{D_{\text{avg}}}{D_{\text{now}}} \quad \text{and} \quad d_{\text{vol}} = \frac{D_{\text{vol}}}{D_{\text{now}}}.$$

If we have two positive securities, D and M, we can find the covariance between their returns: The covariance between the two returns is given by

$$\begin{aligned} \text{Cov}_{m,d} &= \frac{1}{2}\,(m_{\text{up}} - m_{\text{avg}})(s_{\text{up}} - s_{\text{avg}}) + \frac{1}{2}\,(m_{\text{down}} - m_{\text{avg}})(s_{\text{down}} - s_{\text{avg}}) \\ &= m_{\text{vol}}\, d_{\text{vol}}. \end{aligned}$$

We want to write the expected change (*i.e.*, average change) in the asset's value, $D_{\text{avg}} - D_{\text{now}}$, in terms of its risk, D_{vol}. What we know so far is given by (4.6). What we need to know now is the relationship between D_{avg} and $\hat{D}_{\text{avg}}$. The difference between the two turns out to be the *risk premium.* We can see as follows:

$$
\begin{aligned}
D_{\text{avg}} - \hat{D}_{\text{avg}} &= \left(\frac{1}{2} - \hat{p}_{\text{up}}\right) D_{\text{up}} + \left(\frac{1}{2} - \hat{p}_{\text{down}}\right) D_{\text{down}} \\
&= \left(\frac{1}{2} - \hat{p}_{\text{up}}\right) D_{\text{up}} + \left(\hat{p}_{\text{up}} - \frac{1}{2}\right) D_{\text{down}} \\
&= \left(\frac{1}{2} - \hat{p}_{\text{up}}\right) (D_{\text{up}} - D_{\text{down}}) \\
&= (1 - 2\,\hat{p}_{\text{up}}) \left(\frac{D_{\text{up}} - D_{\text{down}}}{2}\right) \\
&= \lambda\, D_{\text{vol}},
\end{aligned}
\tag{4.10a}
$$

where

$$\lambda = 1 - 2\,\hat{p}_{\text{up}} \tag{4.10b}$$

is the *price of risk.* (We will give some meaning to this phrase below.) Notice that λ does not depend on D_{now}, D_{up}, or D_{down} in the following sense: In a complete market with no arbitrage opportunities, we will get the same value for λ no matter which derivative security we choose.

Combining (4.6) and (4.10) produces the result we are looking for:

$$D_{\text{avg}} - D_{\text{now}} = r\, D_{\text{now}} + \lambda\, D_{\text{vol}}. \tag{4.11}$$

Equation (4.11) says that the average change in the value of an asset—any asset including the stock and the bond—equals the sum of two parts: One part is a reward for waiting, $r\, D_{\text{now}}$, and the other part is a reward for bearing risk, $\lambda\, D_{\text{vol}}$. For an asset with zero value right now (for example, the value of a forward contract, which as we saw above, is zero at its inception), the expected change in the asset's value equals its risk premium. For assets with a positive value today, we can write (4.11) as

$$d_{\text{avg}} = r + \lambda\, d_{\text{vol}}. \tag{4.12}$$

Solving (4.12) for λ, we see that the price of risk is the *risk premium per unit of risk*:

$$\lambda = \frac{d_{\text{avg}} - r}{d_{\text{vol}}}.$$

The Radon–Nikodym derivative. What's the relationship between the real probabilities and the pseudo probabilities, $\hat{p}_{\text{up}}$ and $\hat{p}_{\text{down}}$? Since $\hat{p}_{\text{down}} = 1 - \hat{p}_{\text{up}}$, we focus on $\hat{p}_{\text{up}}$. We can solve (4.10b) for $\hat{p}_{\text{up}}$:

$$\hat{p}_{\text{up}} = \frac{1}{2} - \frac{\lambda}{2}. \tag{4.13}$$

Since we must have $0 < \hat{p}_{\text{up}} < 1$, this means that we must have $-1 < \lambda < 1$. (In a more general setting, λ would not be restricted to this range.) We see that if $\lambda = 0$, then $\hat{p}_{\text{up}} = \frac{1}{2}$, the real world probability. The ratio of pseudo probabilities to real probabilities is known as the *Radon–Nikodym derivative*. Using the expression for λ, we can write these ratios as

$$\left(\frac{d\mathcal{Q}}{d\mathcal{P}}\right)_{\text{up}} = \frac{\hat{p}_{\text{up}}}{p_{\text{up}}} = \frac{\hat{p}_{\text{up}}}{1/2} = 1 - \lambda$$
$$\left(\frac{d\mathcal{Q}}{d\mathcal{P}}\right)_{\text{down}} = \frac{\hat{p}_{\text{down}}}{p_{\text{down}}} = \frac{1-\hat{p}_{\text{up}}}{1/2} = 1 + \lambda.$$

We see that

$$\left(\frac{d\mathcal{Q}}{d\mathcal{P}}\right)_{\text{avg}} = 1 \quad \text{and} \quad \left(\frac{d\mathcal{Q}}{d\mathcal{P}}\right)_{\text{vol}} = -\lambda.$$

Note that since the probability that we are where we are right now is one regardless of how we choose to measure the probabilities of future outcomes, we have $\left(\frac{d\mathcal{Q}}{d\mathcal{P}}\right)_{\text{now}} = 1$. So we see that $\left(\frac{d\mathcal{Q}}{d\mathcal{P}}\right)$ is a martingale with (relative) volatility equal to the negative of the price of risk:

$$\left(\frac{d\mathcal{Q}}{d\mathcal{P}}\right)_{\text{avg}} - \left(\frac{d\mathcal{Q}}{d\mathcal{P}}\right)_{\text{now}} = 0 \quad \text{and} \quad \frac{\left(\frac{d\mathcal{Q}}{d\mathcal{P}}\right)_{\text{vol}}}{\left(\frac{d\mathcal{Q}}{d\mathcal{P}}\right)_{\text{now}}} = -\lambda.$$

State–price beta models and the deflator asset. We will consider a special asset, that has a positive value right now, $M_{\text{now}} > 0$. We can consider its returns, m_{up} and m_{down}. Now suppose that the volatility of its return equals the price of risk, $m_{\text{vol}} = \lambda$. Equation (4.12) says that for this asset $m_{\text{avg}} = r + \lambda^2$, where λ^2 is the variance of this return for this asset. We can solve these two equations for m_{up} and m_{down}:

$$m_{\text{up}} = r + \lambda^2 + \lambda \quad \text{and} \quad m_{\text{down}} = r + \lambda^2 - \lambda.$$

We call this special asset the *deflator asset*.

Using $m_{\text{avg}} = r + \lambda^2$, we can write

$$\begin{aligned} \lambda\, d_{\text{vol}} &= \lambda\, d_{\text{vol}} \left(\frac{m_{\text{avg}} - r}{\lambda^2}\right) \\ &= \left(\frac{\lambda\, d_{\text{vol}}}{\lambda^2}\right)(m_{\text{avg}} - r) \\ &= \beta_d \,(m_{\text{avg}} - r)\,, \end{aligned} \tag{4.14}$$

where

$$\beta_d = \frac{\lambda\, d_{\text{vol}}}{\lambda^2}.$$

Combining (4.12) and (4.14) produces

$$d_{\text{avg}} - r = \beta_d \,(m_{\text{avg}} - r)\,. \tag{4.15}$$

Note that $\lambda\, d_{\text{vol}}$ is the covariance between the asset's returns and the deflator asset's returns. What all of this says is that the absence of arbitrage implies that the excess expected return for an asset (*i.e.*, its risk premium), $d_{\text{avg}} - r$, is "determined" by the covariance of the asset's returns with this special asset (scaled by the variance of the special asset).

One of the drawbacks of our simple setup is that we cannot make the covariance between the returns on two assets zero without making one or both of the standard deviations zero. In a richer setting where we can do that, it is possible to have assets that are quite risky in the sense that their standard deviations are large, but have no risk premium because their returns are uncorrelated with the deflator asset. Gold mining stocks are a classic example of such an asset: large standard deviation of returns, small risk premium.[9]

Under some conditions the special asset will turn out to be the *market portfolio*, which has often been interpreted to mean something like the NYSE index. In this case (4.14) is the famous Capital Asset Pricing Model (CAPM), for which Sharpe and Lintner won the Nobel price:

$$d_{\text{avg}} - r = \beta_d \, (m_{\text{avg}} - r) \, . \tag{4.16}$$

Numeraire invariance, trading gains, and the state–price deflator. In this section we consider changing the units in which prices and payoffs are measured.

A *numeraire* is a unit of measurement. For example, we could measure prices and payoffs in terms of U.S. dollars or in terms of French francs or in terms of bushels of wheat. The only thing we require is that the value of the numeraire always stay positive. The *numeraire invariance theorem* says a trading strategy is an arbitrage in terms of one numeraire (one way of measuring things) *if and only if* it is an arbitrage in terms of any other numeraire. This means that if there are no arbitrage opportunities in one numeraire, then there are no arbitrage opportunities in any numeraire. We will look for a numeraire that makes it easy to verify that there are no arbitrage opportunities.

In order to change the numeraire, we use the *exchange rate* between the old and new numeraires. (Technically, this is called a *deflator*.) For example, if we wished to change from dollars to francs, we would use the dollar–franc exchange rate to convert. Or if we wished to change from dollars to bushels of wheat, we would use the number of bushels of wheat per dollar. Let us use an exchange rate to change from one (arbitrary) numeraire to another (arbitrary) numeraire. Let the value of this exchange rate now and next year be given by $Y_{\text{now}} > 0$, $Y_{\text{up}} > 0$, and $Y_{\text{down}} > 0$. In terms of the new units, the value of the asset and its payoffs are given by

$$D^Y_{\text{now}} = Y_{\text{now}} \, D_{\text{now}}, \quad D^Y_{\text{up}} = Y_{\text{up}} \, D_{\text{up}}, \quad \text{and} \quad D^Y_{\text{down}} = Y_{\text{down}} \, D_{\text{down}}.$$

Let's check to see that using a deflator to change the numeraire doesn't affect whether a trading strategy is an arbitrage or not. Suppose we had a trading strategy that cost nothing right now, so that $D_{\text{now}} = 0$; after deflation $D^Y_{\text{now}} = 0$ too. If all the payoffs are positive (or negative) before deflation, then they are all positive (or negative) after deflation. The point is that changing the units in which asset prices and their payoffs are measured doesn't change the *sign* of anything. And since arbitrages are essentially about signs, changing the units doesn't make arbitrage opportunities either appear or disappear.

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[9] An appendix with three states, two risky assets, and a bond?

We can state the conditions for the absence of arbitrage opportunities in terms of *average deflated trading gains*, which are defined as follows:

$$D^Y_{\text{avg}} - D^Y_{\text{now}},$$

where $D^Y_{\text{avg}} = (D^Y_{\text{up}} + D^Y_{\text{down}})/2$ (using the true probabilities). If there is a numeraire that makes the average deflated trading gains equal zero (denote it Y^*),

$$D^{Y^*}_{\text{avg}} - D^{Y^*}_{\text{now}} = 0, \tag{4.17}$$

for all assets, then there are no arbitrage opportunities. The exchange rate that does the trick, Y^*, is called the *state–price deflator*. This way of stating the absence-of-arbitrage condition focuses directly on the definition of an arbitrage. It says that if you pay nothing today (so that $D^{Y^*}_{\text{now}} = 0$) then on average you get nothing back, which means that if you get positive payoffs in some states of the world, you must be getting some negative payoffs too—hence no arbitrage. It also says that if you form a portfolio that produces a positive amount today (which means that the price of the portfolio is negative), then on average you will have to make payments next year—hence no arbitrage.

We may wonder just what units are these that make trading gains martingales and ensure the absence of arbitrage opportunities. As I will discuss below in the section on equilibrium pricing, we can always interpret these as units of "additional happiness."

Let's examine this version of the absence-of-arbitrage condition, equation (4.17):

$$Y^*_{\text{now}}\, D_{\text{now}} = p_{\text{up}}\, (Y^*_{\text{up}}\, D_{\text{up}}) + p_{\text{down}}\, (Y^*_{\text{down}}\, D_{\text{down}}),$$

which we can solve for D_{now}:

$$D_{\text{now}} = \left\{ \frac{1}{2} \left(\frac{Y^*_{\text{up}}}{Y^*_{\text{now}}} \right) \right\} D_{\text{up}} + \left\{ \frac{1}{2} \left(\frac{Y^*_{\text{down}}}{Y^*_{\text{now}}} \right) \right\} D_{\text{down}}. \tag{4.18}$$

Comparing this expression with (4.5), we see that

$$\frac{Y^*_{\text{up}}}{Y^*_{\text{now}}} = \frac{1}{1+r} \left(\frac{\hat{p}_{\text{up}}}{1/2} \right) \quad \text{and} \quad \frac{Y^*_{\text{down}}}{Y^*_{\text{now}}} = \frac{1}{1+r} \left(\frac{1-\hat{p}_{\text{up}}}{1/2} \right).$$

Let's examine the average "return," y^*_{avg}, and volatility of the "return," y^*_{vol}, for the state–price deflator:

$$y^*_{\text{avg}} = -r \left(\frac{1}{1+r} \right) \quad \text{and} \quad y^*_{\text{vol}} = -\lambda \left(\frac{1}{1+r} \right). \tag{4.19}$$

In a continuous-time setting, these expressions simplify to $-r$ and $-\lambda$ respectively. In addition, in continuous time, the deflator asset equals the inverse of the state–price deflator, $M = 1/Y^*$. Instead, in our simple setup the expected return and volatility of the inverse of the state–price deflator are, respectively,

$$\frac{r+\lambda^2}{1-\lambda^2} \quad \text{and} \quad \frac{\lambda\,(1+r)}{1-\lambda^2}.$$

We see in (4.19) that the dynamics of the state–price deflator depend only on the short rate, r, and the price of risk, λ. This is quite general. Moreover, this suggests a powerful and flexible modeling strategy: Assume the existence of the state–price deflator and model the dynamics of the short rate and the price of risk. One is free to model them any way one chooses, subject only to the existence of the solution to the stochastic (in continuous time: differential) equation.

[Be more specific about (4.19). Relate this to the stochastic process appendix. Point the following out: This is the most powerful representation to arise in asset pricing. This *is* the essence of asset pricing.]

Markovian setting. Now let's model the explicit dependence of an asset's payoffs on the value of the state variables. In other words, we will think of D_{now} as an unknown function of state variables. The absence-of-arbitrage condition becomes a PDE. There are rules that tell us how D_{avg} and D_{dev} depend on S an its dynamics.

5. Equilibrium asset pricing

Finance starts with asset prices and focuses on the conditions for the absence of arbitrage opportunities. By contrast, economics starts with individuals who are both investors and consumers and focuses on their optimizing behavior as characterized by "marginal cost equals marginal benefit." It turns out that these two approaches have much in common. In particular, the state–price deflator that we discussed above (*i.e.*, the deflator that guarantees the absence of arbitrage) can be interpreted as the marginal utility of a representative investor/consumer.

An investor/consumer must decide how much to consume, how much to save, and how to invest the savings. Those decisions will depend on the investor/consumers' *preferences*: their patience and their attitudes toward risk and return. Those attitudes are characterized by what economists call a *utility function.* There are three important features of a representative investor/consumer's preferences.

1. The investor/consumer places a higher value on consumption right now than an equal amount of risk-free consumption next year (*i.e.*, the investor/consumer *discounts the future*).
2. The investor/consumer obeys the law of demand with respect to consumption today versus consumption in the future in that he would increase savings in response to an increase in the (real) interest rate and vice-versa (*i.e.*, the investor/consumer has a positive *elasticity of intertemporal substitution*).
3. The investor/consumer prefers to receive the average payoff with certainty rather than gamble on risky outcomes (*i.e.*, the investor/consumer is *risk averse*).

Utility is derived from consumption—both current consumption and expected future consumption. Thus, maximizing consumption today will not maximize utility today: Future prospects affect happiness today. An investor/consumer is constantly choosing how much to consume today and how much to save in order to produce the pattern of consumption through time that maximizes utility. Asset prices are determined by these decisions. To see how this happens, suppose an investor/consumer buys an asset right now and plans to sell it next year. If he did not buy the asset, he could consume more right now, contributing to his utility today. When he buys an asset, he loses this utility. On the other hand, if he buys the asset, he can sell it next year and, depending on what the asset is worth next year, he can increase his consumption by that amount. This prospective increase in future consumption will contribute to his sense of well-being today: It will increase his utility by some amount right now. In order to maximize his overall utility from the perspective of right now, an investor/consumer must balance these two sources of additional utility.

Let $\mathcal{MU}_{\text{now}}$ represent the additional utility an investor/consumer would get right now from a little more consumption right now. Let $\mathcal{MU}_{\text{up}}$ and $\mathcal{MU}_{\text{down}}$ represent the additional utility he would get right now from the prospect of a little more consumption next year in each of the two possible states of the world. (The "up" and "down" subscripts simply distinguish the two states; they do not refer to how the investor/consumer feels about consumption in those states. We discuss good and bad states of the world below.) The additional utility right now from additional consumption next year can be decomposed into the product of three factors. One factor measures the additional utility in a given state next year from additional consumption—conditional on being in that state next year. But because consumption in the future is not as good as consumption today (see factor 1. above), another factor measures the percentage reduction in that additional utility to *discount* it back to right now. The third factor is the probability that the state actually occurs.

Consider an asset with price D_{now} and payoffs D_{up} and D_{down}. The utility cost right now of buying the asset right now is

$$\mathcal{MU}_{\text{now}}\, D_{\text{now}}. \tag{5.1}$$

The utility gain right now in anticipation of selling the asset next year is

$$\mathcal{MU}_{\text{up}}\, D_{\text{up}} + \mathcal{MU}_{\text{down}}\, D_{\text{down}} \tag{5.2}$$

In equilibrium these to sources of additional utility from (5.1) and (5.2) must be equal; otherwise the investor/consumer could make himself better off by changing his plan. Equating (5.1) and (5.2) and solving for D_{now} produces

$$\begin{aligned} D_{\text{now}} &= \left(\frac{\mathcal{MU}_{\text{up}}}{\mathcal{MU}_{\text{now}}}\right) D_{\text{up}} + \left(\frac{\mathcal{MU}_{\text{down}}}{\mathcal{MU}_{\text{now}}}\right) D_{\text{down}} \\ &= p_{\text{up}} \left(\frac{\mathcal{MU}_{\text{up}}/p_{\text{up}}}{\mathcal{MU}_{\text{now}}}\right) D_{\text{up}} + p_{\text{down}} \left(\frac{\mathcal{MU}_{\text{down}}/p_{\text{down}}}{\mathcal{MU}_{\text{now}}}\right) D_{\text{down}}. \end{aligned} \tag{5.3}$$

Equation (5.3) is a general equilibrium asset pricing formula.

An we noted above, we can decompose $\mathcal{MU}_{\text{up}}$ and $\mathcal{MU}_{\text{down}}$ into three components: (i) the additional utility from consumption in that state of the world when that state occurs, (ii) the probability that state occurs, and (iii) the discount factor. (Of course, $\mathcal{MU}_{\text{now}}$ is composed entirely of component(i).)

Let us focus on component (i). The value of an additional unit of consumption in a given state of the world depends on how much consumption will be available in that state. In bad states of the world, consumption opportunities are low and the value of additional consumption is high. Conversely, consumption is high in good times and the value of additional consumption is low. Therefore a security that pays off in good states of the world is not particularly useful: It contributes to the risk across the different states, and investors require a risk premium to hold it. By contrast, a security that pays off in bad states of the world is very attractive—it is like insurance: It reduces risk across states of the world, and investors will pay a premium to hold it.

If consumption grows rapidly between now and next year, we will end up in a good state where additional consumption is not valued highly. By contrast, if consumption grows slowly (or even negatively) between now and then, we will end up in a bad state where additional consumption is valued quite highly. Thus we can see that securities whose returns are highly correlated with the growth rate of consumption are viewed as risky. For a given covariance between the security's returns and the growth rate of consumption, the more risk averse investor/consumers are, the larger the risk premium required to hold the security. By contrast, securities whose returns are negatively correlated with the growth rate of consumption are not at all risky. (Of course the average security will have to be positively correlated.) This is known as the Consumption-based Capital Asset Pricing Model (C–CAPM).

We now examine more closely the relationship between general equilibrium asset pricing and arbitrage-free asset pricing. Consider applying (5.3) to the risk-free bond that we introduced at in the first section. In this case we have

$$D_{\text{now}} = B_{\text{now}}, \quad D_{\text{up}} = (1+r)\,B_{\text{now}}, \quad \text{and} \quad D_{\text{down}} = (1+r)\,B_{\text{now}}.$$

Inserting these expressions into (5.3), dividing through by B_{now}, and rearranging produces

$$\frac{1}{1+r} = \left(\frac{\mathcal{MU}_{\text{up}}}{\mathcal{MU}_{\text{now}}}\right) + \left(\frac{\mathcal{MU}_{\text{down}}}{\mathcal{MU}_{\text{now}}}\right).$$

We see that the sum of the Arrow–Debreu prices is $1/(1+r)$. If we define

$$\hat{p}_{\text{up}} = \left(\frac{\mathcal{MU}_{\text{up}}}{\mathcal{MU}_{\text{now}}}\right)(1+r)$$

$$\hat{p}_{\text{down}} = \left(\frac{\mathcal{MU}_{\text{down}}}{\mathcal{MU}_{\text{now}}}\right)(1+r),$$

then (5.3) becomes

$$D_{\text{now}} = \frac{\hat{D}_{\text{avg}}}{1+r},$$

which is exactly what we had above for arbitrage-free asset pricing.

An explicit example of a utility function. In this section, we present a utility function that will illustrate the points made above. We will build up the complete utility function in stages.

For the first stage, we consider the utility of consumption in a given state, either right now or in the future, in isolation. We will write the consumer's utility as $u(C)$, which means that the utility depends on the amount of consumption. In a more general setting, we would expect that utility would depend on the mix of things that go into the total consumption. Surely a varied diet is more pleasant than a monotonous one. However we will ignore that aspect for simplicity's sake. We will also ignore the fact that the one's happiness may depend on how much or how little one must work. Since we are not interested in explaining variations in the work week over time, we will ignore this aspect as well.

The first feature we want to build into our utility function is this: More is preferred to less. Thus we require that as C increases, so does $u(C)$. This means that u is increasing in C. Let $C_1 < C_2$. Then, if more is preferred to less, $u(C_2) > u(C_1)$. Let's look at the increase in utility by per unit of increase in consumption:

$$\frac{u(C_2) - u(C_1)}{C_2 - C_1} > 0. \tag{5.4}$$

The ratio in (5.4) is known as *marginal utility.* As long as more is preferred to less, marginal utility is positive. The second feature is known as *diminishing marginal utility.* It means that when you don't have much consumption, a little bit more consumption can increase your happiness quite a bit, but when you already have a lot of consumption, a little bit more consumption will not increase your happiness as much. Let $C_1 < C_2 < C_3$. Diminishing marginal utility means

$$\frac{u(C_3) - u(C_2)}{C_3 - C_2} < \frac{u(C_2) - u(C_1)}{C_2 - C_1}. \tag{5.5}$$

(In terms of calculus, the two features are that the first derivative is everywhere positive and the second derivative is everywhere negative: $u'(C) > 0$ and $u''(C) < 0$.) These two features, which determine the *shape* of the utility function, are all that is important. The absolute level of utility is of no consequence. It's fine for the level of utility to be zero or negative, as long as it increases with consumption at a declining rate. For example, let $v(C) = a + u(C)$, where a is a constant. The behavioral implications of $v(C)$ are identical to those of $u(C)$. In other words, if there were two investor/consumers, one with utility function $u(C)$ and the other with utility function $v(C)$, they would always make the same choices.

Now we consider how an individual feels about risky consumption. Would a consumer prefer a gamble that paid on average the same as a sure thing? The sure thing pays C units, while the gamble pays either $C + \delta$ or $C - \delta$ with equal probabilities. The average payoff of the gamble is C. The utility of the sure thing is $u(C)$. We take the utility of the gamble to be the average utility (not the utility of the average):

$$\frac{u(C + \delta) + u(C - \delta)}{2}.$$

The loss in utility of going from the sure thing the gamble to is

$$\left(\frac{u(C + \delta) + u(C - \delta)}{2}\right) - u(C) = \frac{\delta}{2}\left\{\left(\frac{u(C + \delta) - u(C)}{\delta}\right) - \left(\frac{u(C) - u(C - \delta)}{\delta}\right)\right\} < 0, \tag{5.6}$$

where we know the sign of (5.6) from (5.5), with $C_1 = C - \delta$, $C_2 = C$, and $C_3 = C + \delta$. To measure risk aversion, we can find out how fast the utility loss increases as δ increases (normalized by the marginal utility near C). (In terms of calculus, the measure of risk aversion is $-u''(C)/u'(C)$.)

Now let's consider simultaneously consumption right now and uncertain consumption next year. This utility function embodies all of the features we discussed above. Let

$$\mathcal{U} = u(C_{\text{now}}) + \left\{ \frac{\frac{1}{2}\, u(C_{\text{up}}) + \frac{1}{2}\, u(C_{\text{down}})}{1+\rho} \right\}, \tag{5.7}$$

where $u(\,\cdot\,)$ measures the utility of consumption in a given state of the world and $\mathcal{U}$ measures the utility of the total consumption plan. $\rho > 0$ measures the subjective discount rate that results in valuing future certain consumption less than current consumption. To see this, suppose $C_{\text{now}} = C_{\text{up}} = C_{\text{down}} = C$. Now consider the utility value this consumption right now versus next year:

$$u(C) > \frac{u(C)}{1+\rho}.$$

Next consider a specific functional form for u:

$$u(C) = \begin{cases} \dfrac{C^{1-\gamma} - 1}{1-\gamma} & \gamma \neq 1 \\ \log(C) & \gamma = 1. \end{cases} \tag{5.8}$$

where $\gamma \geq 0$. For this utility function, marginal utility (for small increases in consumption) is given by $C^{-\gamma}$, and the *coefficient of relative risk aversion* is γ. (The measure of risk aversion is $-u''(C)/u'(C) = \gamma/C$. Hence γ measures risk aversion relative to the level of consumption.)

How much additional utility will a bit more consumption deliver for our investor/-consumer?

$$\mathcal{MU}_{\text{now}} = (C_{\text{now}})^{-\gamma}, \quad \mathcal{MU}_{\text{up}} = \frac{\frac{1}{2}\,(C_{\text{up}})^{-\gamma}}{1+\rho}, \quad \text{and} \quad \mathcal{MU}_{\text{down}} = \frac{\frac{1}{2}\,(C_{\text{down}})^{-\gamma}}{1+\rho}. \tag{5.9}$$

We see in (5.9) that we have indeed decomposed $\mathcal{MU}_{\text{up}}$ and $\mathcal{MU}_{\text{down}}$ into the three components discussed above. Let the growth rates of consumption be given by

$$c_{\text{up}} = \frac{C_{\text{up}}}{C_{\text{now}}} - 1 \quad \text{and} \quad c_{\text{down}} = \frac{C_{\text{down}}}{C_{\text{now}}} - 1.$$

The intertemporal marginal rates of substitution are given by

$$\frac{\mathcal{MU}_{\text{up}}}{\mathcal{MU}_{\text{now}}} = \frac{\frac{1}{2}\,(1+c_{\text{up}})^{-\gamma}}{1+\rho} \quad \text{and} \quad \frac{\mathcal{MU}_{\text{down}}}{\mathcal{MU}_{\text{now}}} = \frac{\frac{1}{2}\,(1+c_{\text{down}})^{-\gamma}}{1+\rho}. \tag{5.10}$$

Recalling that the sum of the state prices is $1/(1+r)$, we can write

$$\frac{1}{1+r} = \frac{\frac{1}{2}\,(1+c_{\text{up}})^{-\gamma} + \frac{1}{2}\,(1+c_{\text{down}})^{-\gamma}}{1+\rho}. \tag{5.11}$$

If the growth rate of consumption were state-independent so that $c_{\text{up}} = c_{\text{down}} = c_{\text{avg}}$, then we could write (5.11) as

$$1 + r = (1+\rho)\,(1+c_{\text{avg}})^{\gamma}. \tag{5.12}$$

If consumption for example were constant ($c_{\text{avg}} = 0$), then $r = \rho$. We can take logs of (5.12) and use the approximation $\log(1 + x) \approx x$ for small x to obtain

$$r \approx \rho + \gamma\, c_{\text{avg}}. \tag{5.13}$$

We can see how the equilibrium interest rate depends on the rate of time preference and the expected growth rate of consumption. We can solve (5.13) for c_{avg}:

$$c = \frac{r - \rho}{\gamma}.$$

We see that an increase in the interest rate will increase the growth rate of consumption by $1/\gamma$, the elasticity of intertemporal substitution.

Given (5.10), we can write the valuation formula as

$$D_{\text{now}} = \frac{1}{2}\left(\frac{(1 + c_{\text{up}})^{-\gamma}}{1 + \rho}\right) D_{\text{up}} + \frac{1}{2}\left(\frac{(1 + c_{\text{down}})^{-\gamma}}{1 + \rho}\right) D_{\text{down}}. \tag{5.14}$$

Comparing (5.14) with (4.18), we see that the state–price deflator can be written in terms of the discounted marginal utility of consumption:

$$\frac{Y^*_{\text{up}}}{Y^*_{\text{now}}} = \frac{(1 + c_{\text{up}})^{-\gamma}}{1 + \rho} \quad \text{and} \quad \frac{Y^*_{\text{down}}}{Y^*_{\text{now}}} = \frac{(1 + c_{\text{down}})^{-\gamma}}{1 + \rho}.$$

Therefore we have

$$y^*_{\text{vol}} = \frac{(1 + c_{\text{up}})^{-\gamma} - (1 + c_{\text{down}})^{-\gamma}}{2\,(1 + \rho)}.$$

Above, we found an expression relating y^*_{vol} and λ, (4.19), which together with (5.11) delivers an expression for λ:

$$\lambda = -(1 + r)\, y^*_{\text{vol}} = \frac{(1 + c_{\text{down}})^{-\gamma} - (1 + c_{\text{up}})^{-\gamma}}{(1 + c_{\text{up}})^{-\gamma} + (1 + c_{\text{down}})^{-\gamma}} \approx \frac{\gamma\, c_{\text{vol}}}{1 + c_{\text{avg}}}. \tag{5.15}$$

(In continuous time, we get $\lambda = \gamma\, c_{\text{vol}}$ exactly.) This relation captures the problem this model has: We need a big λ, but c_{vol} is small and therefore γ must be quite big—too big.

Wealth. We may now ask about what we have so far ignored: What is the source of the consumption? How is the consumption is produced? However that may be, we can find the value of the source of consumption by treating consumption as the dividends (*i.e.*, payoffs) that accrues to that source:

$$\begin{aligned} W_{\text{now}} &= \frac{1}{2}\left(\frac{(1 + c_{\text{up}})^{-\gamma}}{1 + \rho}\right) C_{\text{up}} + \frac{1}{2}\left(\frac{(1 + c_{\text{down}})^{-\gamma}}{1 + \rho}\right) C_{\text{down}} \\ &= \left(\frac{\frac{1}{2}\,(1 + c_{\text{up}})^{1-\gamma} + \frac{1}{2}\,(1 + c_{\text{down}})^{1-\gamma}}{1 + \rho}\right) C_{\text{now}}. \end{aligned}$$

This is the *ex dividend* value of the investor/consumer's wealth: It does not include the value of current consumption. Notice that when $\gamma = 1$, the capital–consumption ratio depends only on the rate of time preference, ρ:

$$\frac{W_{\text{now}}}{C_{\text{now}}} = \frac{1}{1+\rho}.$$

6. The term structure of interest rates

In this section we take the setup we have been using and reinterpret the bond and the stock as one- and two-period default-free zero-coupon bonds. The one-period bond pays off one unit in both states next year. Given the one-period interest rate today, r_{now}, the price of the one-period bond today,

$$B_{1,\text{now}} = \frac{1}{1+r_{1,\text{now}}}. \tag{6.1}$$

The payoffs for the two period bond next year are the bond's price. Next year, the two-year bond will become a one-year bond. It's price will depend on the one-year interest rate next year. We now have a problem of terminology with using "up" and "down". If the one-year interest rate is high next year, the one-year bond price will be low (and vice versa). We choose to associate "up" and "down" with the interest rate. With this terminology we have

$$B_{1,\text{up}} = \frac{1}{1+r_{1,\text{up}}} \quad \text{and} \quad B_{1,\text{down}} = \frac{1}{1+r_{1,\text{down}}}. \tag{6.2}$$

One "problem" with this choice arrises. Previously, the price of risk was associated with an *asset price.* A positive price of risk meant that an asset was risky (rather than being insurance). But in this case, the price of risk is associated with movements of the interest rate. Since bond prices (which are asset prices) move inversely with interest rates, the ups and downs of the interest rate are associated with the downs and ups of bond prices. Therefore, if bonds are risky assets (rather than being insurance), then the price of risk must be *negative.* In other words, a negative price of *interest rate risk* implies a positive price of *bond price risk.*

We can get a lot of mileage out a very simple model of the dynamics of the short-term interest rate. Let

$$r_{1,\text{up}} = r_{1,\text{now}} + \delta \quad \text{and} \quad r_{1,\text{down}} = r_{1,\text{now}} - \delta,$$

where $\delta > 0$. Therefore we have

$$r_{1,\text{avg}} = r_{1,\text{now}} \quad \text{and} \quad r_{1,\text{vol}} = \delta.$$

In this model, the short rate is a martingale, since $r_{1,\text{avg}} - r_{1,\text{now}} = 0$. Using (6.2), we can write

$$B_{2,\text{avg}} = \frac{B_{1,\text{up}} + B_{1,\text{down}}}{2} = \frac{\frac{1}{1+r_{1,\text{up}}} + \frac{1}{1+r_{1,\text{down}}}}{2} = \frac{1+r_{1,\text{now}}}{(1+r_{1,\text{now}})^2 - \delta^2}$$

and

$$B_{2,\text{vol}} = \frac{B_{1,\text{up}} - B_{1,\text{down}}}{2} = \frac{\frac{1}{1+r_{1,\text{up}}} - \frac{1}{1+r_{1,\text{down}}}}{2} = \frac{-\delta}{(1+r_{1,\text{now}})^2 - \delta^2}.$$

Let the price of the two-year bond right now be denoted $B_{2,\text{now}}$. Having completely specified the short-term interest rate (now and in all future states), can we say what the price of a two-year bond is right now? The answer is no. The only thing we can say is that one bond does not dominate the other. From (2.2) and (6.1), we can write the no-dominance condition as

$$B_{1,\text{up}} < \frac{B_{2,\text{now}}}{B_{1,\text{now}}} < B_{1,\text{down}}, \tag{6.3}$$

where $B_{2,\text{now}}/B_{1,\text{now}}$ is the forward price of the two-year bond. Within these bounds, we have a degree of freedom left in pinning down the price of the two-year bond today. This degree of freedom is captured by the price of risk: In combination with the dynamics of the short-term interest rate, the price of risk will pin down the price of the long-term bond. We can solve the expression for the relation between risk and return (4.11) for the price of the two-period bond

$$B_{2,\text{now}} = \frac{B_{2,\text{avg}} - \lambda\, B_{2,\text{vol}}}{1 + r_{1,\text{now}}} = \frac{1 + \frac{\lambda\,\delta}{1+r_{1,\text{now}}}}{(1 + r_{1,\text{now}})^2 - \delta^2}. \tag{6.4}$$

Spot yields and forward rates. The two-year yield (compounded annually is defined implicitly in

$$B_{2,\text{now}} = \frac{1}{(1 + r_{2,\text{now}})^2}. \tag{6.5}$$

We can solve (6.5) for $r_{2,\text{now}}$:

$$r_{2,\text{now}} = \sqrt{\frac{1}{B_{2,\text{now}}}} - 1. \tag{6.6}$$

In our case, the two-year yield is

$$r_{2,\text{now}} = \sqrt{\frac{(1 + r_{1,\text{now}})^2 - \delta^2}{1 + \frac{\lambda\,\delta}{1+r_{1,\text{now}}}}} - 1.$$

The forward price of the two-year bond is

$$K_F = (1 + r_{1,\text{now}})\, B_{2,\text{now}} = B_{2,\text{avg}} - \lambda\, B_{2,\text{vol}} = \frac{1 + r_{1,\text{now}} + \lambda\,\delta}{(1 + r_{1,\text{now}})^2 - \delta^2}.$$

The *forward rate* is defined implicitly in the following equation:

$$K_F = \frac{1}{1 + F_{1,\text{now}}}.$$

Therefore, the forward rate is

$$F_{1,\text{now}} = \frac{(1 + r_{1,\text{now}})^2 - \delta^2}{1 + r_{1,\text{now}} + \lambda\,\delta} - 1. \tag{6.7}$$

We focus on now on the relation between the forward rate and the expected future short rate, $r_{1,\text{avg}}$ (which in this example equals $r_{1,\text{now}}$). The difference between the forward rate and the expected future spot rate is called the *forward premium*. We can write the forward premium as

$$F_{1,\text{now}} - r_{1,\text{avg}} = \frac{-\delta\left(\delta + \lambda\,(1 + r_{1,\text{now}})\right)}{1 + r_{1,\text{now}} + \lambda\,\delta}. \tag{6.8}$$

Clearly, from (6.8) we see that if $\delta = 0$, $F_{1,\text{now}} = r_{1,\text{avg}}$. But if $\delta > 0$, then there are two competing forces that drive a wedge between the forward rate and the expected future spot rate. First, if the price of risk were zero, then the remaining force is the *convexity effect*, due to *Jensen's inequality*:

$$F_{1,\text{now}} - r_{1,\text{avg}} = \frac{-\delta^2}{1 + r_{1,\text{now}}},$$

where δ^2 is the variance of the short rate. Therefore, if the price of risk were zero, the yield curve would slope downward on average.

However, the yield curve slopes upward on average. The other effect is due to the risk premium, $-\lambda\,\delta$. The sign of the net effect is given by the sign of

$$\delta + \lambda\,(1 + r_{1,\text{now}}).$$

If the sign of this expression is negative, then $F_{1,\text{now}} - r_{1,\text{avg}} > 0$. Roughly speaking, as long as $-\lambda > \delta$, the forward rate will be above the expected future spot rate and the yield curve will slope upward on average.

A three-year bond. Let us extend our term structure model by another year. We denote the price right now of a three-year, default-free zero coupon bond by $B_{3,\text{now}}$. Next year, this bond will become a two-year bond. Therefore, if we extend our model of the dynamics of the short rate one more period, we can apply our pricing formula for the two-year bond today to the three-year bond next year. Then we can take those two values for the two-year bond as payoffs next year to find the value of the three-year bond right now. This recursive structure can be extended indefinitely.

Here is how we extend our model of the interest rate. If the interest rate turns out to be $r_{1,\text{up}}$ next year, then the interest rate the following year will be either

$$r_{1,\text{up}} + \delta = r_{1,\text{now}} + 2\,\delta \quad \text{or} \quad r_{1,\text{up}} - \delta = r_{1,\text{now}}$$

with equal probabilities, and if the interest rate turns out to be $r_{1,\text{down}}$ next year, then the interest rate the following year will be either

$$r_{1,\text{down}} + \delta = r_{1,\text{now}} \quad \text{or} \quad r_{1,\text{down}} - \delta = r_{1,\text{now}} - 2\,\delta$$

with equal probabilities. The structure of possible rates through time looks like this:

$$r_{1,\text{now}} \begin{cases} r_{1,\text{now}} + \delta & \begin{cases} r_{1,\text{now}} + 2\,\delta \\ \\ r_{1,\text{now}} \end{cases} \\ \\ r_{1,\text{now}} - \delta & \begin{cases} r_{1,\text{now}} \\ \\ r_{1,\text{now}} - 2\,\delta \end{cases} \end{cases} \tag{6.9}$$

With this setup, next year $r_{1,\text{up}} = r_{1,\text{now}} + \delta$ and $r_{1,\text{down}} = r_{1,\text{now}} - \delta$ play the role that $r_{1,\text{now}}$ plays right now. We can stick these one-period interest rates into the two-year bond pricing formula (6.4) to find the price of a two-year bond in each state next year:

$$B_{2,\text{up}} = \frac{1 + \frac{\lambda\,\delta}{1+r_{1,\text{up}}}}{(1 + r_{1,\text{up}})^2 - \delta^2}$$
$$B_{2,\text{down}} = \frac{1 + \frac{\lambda\,\delta}{1+r_{1,\text{down}}}}{(1 + r_{1,\text{down}})^2 - \delta^2}.$$

We can find the price right now for the three-year bond by applying the formula:

$$B_{3,\text{now}} = \frac{B_{3,\text{avg}} - \lambda\,B_{3,\text{vol}}}{1 + r_{1,\text{now}}}.$$

Although this procedure is straightforward, it is beginning to get a bit messy.

The forward price of the three-year bond for delivery in two years (at which time it will have become a one-year bond) is

$$\frac{B_{3,\text{now}}}{B_{2,\text{now}}} = \frac{B_{3,\text{avg}} - \lambda\,B_{3,\text{vol}}}{B_{2,\text{avg}} - \lambda\,B_{2,\text{vol}}}.$$

The forward rate associated with this forward price is

$$F_{3,\text{now}} = \frac{B_{2,\text{avg}} - \lambda\,B_{2,\text{vol}}}{B_{3,\text{avg}} - \lambda\,B_{3,\text{vol}}} - 1,$$

and, since the average short-term interest rate two years from now, $r_{1,\text{avg}(2)}$, equals $r_{1,\text{now}}$ from today's perspective, the forward premium is

$$\begin{aligned} F_{3,\text{now}} - r_{1,\text{avg}(2)} &= \frac{B_{2,\text{avg}} - \lambda\,B_{2,\text{vol}}}{B_{3,\text{avg}} - \lambda\,B_{3,\text{vol}}} - (1 + r_{1,\text{now}}) \\ &= \frac{-3\,\delta^2\,(1 + r_{1,\text{now}})\,\left(1 + \lambda^2\right) - 2\,\delta\,\lambda\,\left((1 + r_{1,\text{now}})^2 + 2\,\delta^2\right)}{\left((1 + r_{1,\text{now}})^2 - \delta^2\right) + 3\,\delta\,\lambda\,(1 + r_{1,\text{now}} + \delta\,\lambda)}. \end{aligned} \tag{6.10}$$

The two terms in the numerator have opposite signs (assuming the price of risk is negative). The first term is the convexity term and the second term is the risk premium. If the price of risk were zero, (6.10) simplifies to

$$F_{3,\text{now}} - r_{1,\text{avg}(2)} = \frac{-3\,\delta^2\,(1 + r_{1,\text{now}})}{(1 + r_{1,\text{now}})^2 - \delta^2} < 0.$$

Also note that when $\lambda = 0$, the forward rates get progressively lower:

$$F_{3,\text{now}} < F_{2,\text{now}}.$$

Which pseudo probabilities? In the previous section, we let the one-period risk-free interest rate change randomly through time. In this section, we continue to explore some of the effects of randomly changing interest rates. We will use the information we have to uncover the Arrow–Debreu prices for four states of the world two years from now. (Thus far, we have only priced default-free bonds that payoff the same amount in every state.) Even though the state prices are uniquely determined, we will see that there are two ways to factor them pseudo probabilities and present values. Also we will see which of those pseudo probability distributions can be extracted using option prices and bond prices.

We adopt the setup from the previous section. Refer to the tree-like structure in (6.9). Two years from now, there will be three possible states of the world:

$$r_{1,\text{now}} - 2\,\delta, \quad r_{1,\text{now}}, \quad \text{and} \quad r_{1,\text{now}} + 2\,\delta.$$

On the other hand, there are four possible *paths* or sequences of the short rate, two of which end up at $r_{1,\text{now}}$:

$$\begin{array}{lllll}
(i) & r_{1,\text{now}} & \xrightarrow{\text{up}} & r_{1,\text{now}} + \delta & \xrightarrow{\text{up}} & r_{1,\text{now}} + 2\,\delta \\
(ii) & r_{1,\text{now}} & \xrightarrow{\text{up}} & r_{1,\text{now}} + \delta & \xrightarrow{\text{down}} & r_{1,\text{now}} \\
(iii) & r_{1,\text{now}} & \xrightarrow{\text{down}} & r_{1,\text{now}} - \delta & \xrightarrow{\text{up}} & r_{1,\text{now}} \\
(iv) & r_{1,\text{now}} & \xrightarrow{\text{down}} & r_{1,\text{now}} - \delta & \xrightarrow{\text{down}} & r_{1,\text{now}} - 2\,\delta.
\end{array}$$

Each of these paths is equally likely with probability $\frac{1}{4}$. Therefore, by adding up the probabilities of the paths that end up at $r_{1,\text{now}}$, we see that the the probability that the short rate will be $r_{1,\text{now}}$ in two years is $\frac{1}{2}$. In addition, we see that the average value of the short rate in two years is $r_{1,\text{now}}$. It turns out, however, that the rates at year two play no role in the analysis that follows. They are simply "placeholders" used to distinguish among states.

At each "node" on the tree where the interest rate branches into two possibilities, now and next year, the pseudo probabilities of the branches are given by

$$\hat{p}_{\text{up}} = \frac{1 - \lambda}{2} \quad \text{and} \quad \hat{p}_{\text{down}} = 1 - \hat{p}_{\text{up}} = \frac{1 + \lambda}{2}.$$

Therefore, the pseudo probability for each path is given by

$$\hat{p}_{\text{up,up}} = \hat{p}_{\text{up}}^2 = \frac{(1-\lambda)^2}{4} \tag{6.11a}$$

$$\hat{p}_{\text{up,down}} = \hat{p}_{\text{up}}\,\hat{p}_{\text{down}} = \frac{(1-\lambda)\,(1+\lambda)}{4} \tag{6.11b}$$

$$\hat{p}_{\text{down,up}} = \hat{p}_{\text{down}}\,\hat{p}_{\text{up}} = \frac{(1+\lambda)\,(1-\lambda)}{4} \tag{6.11c}$$

$$\hat{p}_{\text{down,down}} = \hat{p}_{\text{down}}^2 = \frac{(1+\lambda)^2}{4}. \tag{6.11d}$$

These probabilities add up to one, since the four paths are the only possible ones. The probability that the short rate will be $r_{1,\text{now}}$ in two years is the sum of the probabilities of the two middle paths [(6.11b) plus (6.11c)],

$$2\,\hat{p}_{\text{up}}\,\hat{p}_{\text{down}} = \frac{(1+\lambda)\,(1-\lambda)}{2}. \tag{6.12}$$

Clearly when $\lambda = 0$ these pseudo path probabilities are the same as the physical path probabilities.

Also at each node, the Arrow–Debreu state prices for payoffs one step ahead are given by

$$\frac{\hat{p}_{\text{up}}}{1+r_{1,n}} = \frac{1-\lambda}{2\,(1+r_{1,n})} \quad \text{and} \quad \frac{\hat{p}_{\text{down}}}{1+r_{1,n}} = \frac{1+\lambda}{2\,(1+r_{1,n})},$$

where n is either "now" or "up" or "down," depending on where we are in the tree. The price of a unit payoff at the end of each of the four paths is given by the product of the prices at each node in the path:

$$(i) \qquad \left(\frac{\hat{p}_{\text{up}}}{1+r_{1,\text{now}}}\right)\left(\frac{\hat{p}_{\text{up}}}{1+r_{1,\text{up}}}\right) = \frac{(1-\lambda)^2}{4\,(1+r_{1,\text{now}})\,(1+r_{\text{now}}+\delta)} \tag{6.13a}$$

$$(ii) \qquad \left(\frac{\hat{p}_{\text{up}}}{1+r_{1,\text{now}}}\right)\left(\frac{\hat{p}_{\text{down}}}{1+r_{1,\text{up}}}\right) = \frac{(1-\lambda)\,(1+\lambda)}{4\,(1+r_{1,\text{now}})\,(1+r_{\text{now}}+\delta)} \tag{6.13b}$$

$$(iii) \qquad \left(\frac{\hat{p}_{\text{down}}}{1+r_{1,\text{now}}}\right)\left(\frac{\hat{p}_{\text{up}}}{1+r_{1,\text{down}}}\right) = \frac{(1+\lambda)\,(1-\lambda)}{4\,(1+r_{1,\text{now}})\,(1+r_{\text{now}}-\delta)} \tag{6.13c}$$

$$(iv) \qquad \left(\frac{\hat{p}_{\text{down}}}{1+r_{1,\text{now}}}\right)\left(\frac{\hat{p}_{\text{down}}}{1+r_{1,\text{down}}}\right) = \frac{(1+\lambda)^2}{4\,(1+r_{1,\text{now}})\,(1+r_{\text{now}}-\delta)}. \tag{6.13d}$$

Therefore the price of a unit payoff conditional on the short rate equaling $r_{1,\text{now}}$ in two years is the sum of the two middle prices [(6.13b) plus (6.13c)]:

$$\frac{(1+\lambda)\,(1-\lambda)}{2\,(1+r_{1,\text{now}}+\delta)\,(1+r_{1,\text{now}}-\delta)}. \tag{6.14}$$

We now have the price of a unit payout in each of the three possible states of the world in two years, given by (6.13a), (6.14), and (6.13d). Recall that a two-year default-free zero-coupon bond is a claim to one unit in each state in two years. Indeed, we can confirm that the sum of these three values is the value of the two-year bond, $B_{2,\text{now}}$.

Since the three values are all positive and add up to $B_{2,\text{now}}$, we can divide them each by $B_{2,\text{now}}$ and interpret them as probabilities:

$$\tilde{p}_{2,\text{up}} = \left(\frac{1 + r_{1,\text{now}} - \delta}{1 + r_{1,\text{now}} + \delta\,\lambda}\right)\left(\frac{(1-\lambda)^2}{4}\right) \tag{6.15a}$$

$$\tilde{p}_{2,\text{middle}} = \left(\frac{1 + r_{1,\text{now}}}{1 + r_{1,\text{now}} + \delta\,\lambda}\right)\left(\frac{(1+\lambda)\,(1-\lambda)}{2}\right) \tag{6.15b}$$

$$\tilde{p}_{2,\text{down}} = \left(\frac{1 + r_{1,\text{now}} + \delta}{1 + r_{1,\text{now}} + \delta\,\lambda}\right)\left(\frac{(1+\lambda)^2}{4}\right). \tag{6.15c}$$

Moreover, we can easily compute the price of any asset that has payoffs only two years from now (*i.e.*, with no payoffs one year from now):

$$D_{\text{now}} = B_{2,\text{now}}\left(\tilde{p}_{2,\text{up}}\,D_{2,\text{up}} + \tilde{p}_{2,\text{middle}}\,D_{2,\text{middle}} + \tilde{p}_{2,\text{down}}\,D_{2,\text{down}}\right), \tag{6.16}$$

which is the present value of the average payoff using the probabilities in (6.15). Note, however, that these probabilities are not the same as the pseudo path probabilities we calculated above, (6.11a), (6.12), and (6.11d). Only if $\delta = 0$ are they the same. This corresponds to the case in where the interest rate is not stochastic.

The probabilities will we uncover using option prices turn out to be those in (6.15). Basically, using the prices of options on the short rate itself, we can isolate the value of a unit payoff in any state of the world associated with the short rate, say $B_{2,\text{now}}\,\tilde{p}_{2,\text{down}}$. To get the original pseudo probabilities from this value, we would need to know which paths of the short rate to associate with this state. In our example this is not hard to do, but in a more realistic setting this would not be possible using only bond prices and option prices. What we can extract with these prices (obviously) is

$$\tilde{p}_{2,\text{down}} = \frac{B_{2,\text{now}}\,\tilde{p}_{2,\text{down}}}{B_{2,\text{now}}}.$$

For a specific example, consider a (European) put option on the short-term interest rate that expires two-years from now, where

$$r_{1,\text{now}} - 2\,\delta < K_P < r_{1,\text{now}}.$$

This put ends in the money only if the interest rate ends up at $r_{1,\text{now}} - 2\,\delta$. Therefore, using (6.13d), the value of the put today is

$$P_{\text{now}} = \big(K_P - (r_{1,\text{now}} - 2\,\delta)\big)\left(\frac{(1+\lambda)^2}{4\,(1 + r_{1,\text{now}})\,(1 + r_{\text{now}} - \delta)}\right).$$

Using our trick with two strike prices, both between $r_{1,\text{now}} - 2\,\delta$ and $r_{1,\text{now}}$, we can extract the Arrow–Debreu state price

$$\frac{(1+\lambda)^2}{4\,(1 + r_{1,\text{now}})\,(1 + r_{\text{now}} - \delta)}.$$

However, without knowing all of the paths of the money-market account that lead to that value of the short rate (which, in a more realistic model, we would not know), the only thing we can do is divide this state price by the value of the two-year bond, producing (6.15c) rather than (6.11d).

There is another way we can decompose the price of the two-year bond (or any other asset price). Consider the *money-market account*. The money market account starts with an initial investment of one unit and earns the short-term risk-free interest rate each period, $\beta_{\text{now}} = 1$. Right now, we know exactly what the money-market account will be worth after one year, $\beta_1 = 1 + r_{1,\text{now}}$, but we don't know what it will be worth after two years because we don't know what the short-term interest rate one year from now. In two years the money-market account will be worth either

$$\beta_{2,\text{up}} = (1 + r_{1,\text{now}})\,(1 + r_{1,\text{up}}) \quad \text{or} \quad \beta_{2,\text{down}} = (1 + r_{1,\text{now}})\,(1 + r_{1,\text{down}}).$$

Now we can write the Arrow–Debreu prices for the end of each path (6.13) in terms of the path pseudo probabilities (6.11) and the value of the money-market account:

$$\frac{\hat{p}_{\text{up,up}}}{\beta_{2,\text{up}}}, \quad \frac{\hat{p}_{\text{up,down}}}{\beta_{2,\text{up}}}, \quad \frac{\hat{p}_{\text{down,up}}}{\beta_{2,\text{down}}}, \quad \text{and} \quad \frac{\hat{p}_{\text{down,down}}}{\beta_{2,\text{down}}}.$$

Therefore, the price of an asset right now that has payoffs at the end of each path is

$$\begin{aligned} D_{\text{now}} = \hat{Z}_{2,\text{avg}} = \hat{p}_{\text{up,up}} \left(\frac{D_{\text{up,up}}}{\beta_{2,\text{up}}}\right) + \hat{p}_{\text{up,down}} \left(\frac{D_{\text{up,down}}}{\beta_{2,\text{up}}}\right) \\ + \hat{p}_{\text{down,up}} \left(\frac{D_{\text{down,up}}}{\beta_{2,\text{down}}}\right) + \hat{p}_{\text{down,down}} \left(\frac{D_{\text{down,down}}}{\beta_{2,\text{down}}}\right), \end{aligned}$$

where $Z_2 = \frac{D_2}{\beta_2}$.

The forward price vs. the futures price. When the short rate is stochastic, forward price and futures prices or not the same. In our example, we must look at forward and futures prices for delivery in two years. If we look at forward and futures prices for delivery in one year, they will be the same because the interest rate has no opportunity to change prior to delivery.

Since the payoffs for the forward contract all occur in two years, we can use (6.16) to find the value of K_F that makes the value of the forward contract (on the stock) zero right now:

$$\begin{aligned} 0 = B_{2,\text{now}}\big(\tilde{p}_{2,\text{up}}\,(S + 2\,\epsilon - K_F) \\ + \tilde{p}_{2,\text{middle}}\,(S - K_F) + \tilde{p}_{2,\text{down}}\,(S - 2\,\epsilon - K_F)\big), \end{aligned} \tag{6.17}$$

where ϵ is the shock to the stock price. Solving (6.17) for K_F produces

$$\begin{aligned} K_F &= S + \epsilon\,(\tilde{p}_{2,\text{up}} - \tilde{p}_{2,\text{down}}) \\ &= S - \epsilon \left(\frac{\delta\,(1 + \lambda^2) + 2\,\lambda\,(1 + r_{1,\text{now}})}{1 + r_{1,\text{now}} + \delta\,\lambda}\right). \end{aligned}$$

What is the value next year of a forward contract entered into right now? Next year the forward contract will be an asset that makes its payoffs in one year. Its value will be either

$$\frac{\hat{p}_{\text{up}}\,(S + 2\,\epsilon - K_F) + \hat{p}_{\text{down}}\,(S - K_F)}{1 + r_{1,\text{now}} + \delta} = \frac{\epsilon\,(1 + \lambda)}{1 + r_{1,\text{now}} + \delta\,\lambda}$$

or

$$\frac{\hat{p}_{\text{up}}\,(S - K_F) + \hat{p}_{\text{down}}\,(S - 2\,\epsilon - K_F)}{1 + r_{1,\text{now}} - \delta} = \frac{-\epsilon\,(1-\lambda)}{1 + r_{1,\text{now}} - \delta\,\lambda},$$

depending on which state we end up in.

A futures contract requires *resettlement*: Each period any losses on your futures position must be paid and any gains will be received. This is in contrast to a forward contract for which no settlement is made until the delivery date. This resettlement feature means that a futures contract pays "dividends" prior to the delivery date, and those dividends ensure that immediately after they are paid the value of the futures contract is always zero. Since the gains and losses are computed from changes in the futures price (which is not the value of a futures contract), the dividends must equal those changes. Let $K_{\mathcal{F},\text{now}}$ denote the futures price today. The structure of the dividends for the futures contract looks like this:

$$\begin{cases} K_{\mathcal{F},\text{up}} - K_{\mathcal{F},\text{now}} & \begin{cases} (S + 2\,\epsilon) - K_{\mathcal{F},\text{up}} \\ \\ S - K_{\mathcal{F},\text{up}} \end{cases} \\ \\ K_{\mathcal{F},\text{down}} - K_{\mathcal{F},\text{now}} & \begin{cases} S - K_{\mathcal{F},\text{down}} \\ \\ (S - 2\,\epsilon) - K_{\mathcal{F},\text{down}} \end{cases} \end{cases} \tag{6.18}$$

We can find the futures price today by recursively solving backward from the delivery date. First find the values of $K_{\mathcal{F},\text{up}}$ and $K_{\mathcal{F},\text{down}}$ that make the value of the futures contract equal zero in each of the two states next year; then find the futures price today that makes the value of a claim to the dividends paid next year equal zero.

We start by determining $K_{\mathcal{F},\text{up}}$ and $K_{\mathcal{F},\text{down}}$. The condition that the value of the futures contract be zero in each state next year can be written as follows:

$$\frac{\hat{p}_{\text{up}}\,\big((S + 2\,\epsilon) - K_{\mathcal{F},\text{up}}\big) + \hat{p}_{\text{down}}\,\big(S - K_{\mathcal{F},\text{up}}\big)}{1 + r_{1,\text{up}}} = 0$$

$$\frac{\hat{p}_{\text{up}}\,\big(S - K_{\mathcal{F},\text{down}}\big) + \hat{p}_{\text{down}}\,\big((S - 2\,\epsilon) - K_{\mathcal{F},\text{down}}\big)}{1 + r_{1,\text{down}}} = 0.$$

These conditions can be solved for $K_{\mathcal{F},\text{up}}$ and $K_{\mathcal{F},\text{down}}$:

$$\begin{aligned} K_{\mathcal{F},\text{up}} &= \hat{p}_{\text{up}}\,(S + 2\,\epsilon) + \hat{p}_{\text{down}}\,S \\ K_{\mathcal{F},\text{down}} &= \hat{p}_{\text{up}}\,S + \hat{p}_{\text{down}}\,(S - 2\,\epsilon). \end{aligned} \tag{6.19}$$

Now the futures price today must by chosen to make payoffs equal to the change in the futures price have no value today:

$$\frac{\hat{p}_{\text{up}}\,(K_{\mathcal{F},\text{up}} - K_{\mathcal{F},\text{now}}) + \hat{p}_{\text{down}}\,(K_{\mathcal{F},\text{down}} - K_{\mathcal{F},\text{now}})}{1 + r_{1,\text{now}}} = 0,$$

which we can solve for $K_{\mathcal{F},\text{now}}$:

$$K_{\mathcal{F},\text{now}} = \hat{p}_{\text{up}}\, K_{\mathcal{F},\text{up}} + \hat{p}_{\text{down}}\, K_{\mathcal{F},\text{down}}. \tag{6.20}$$

Now we can substitute in expressions for $K_{\mathcal{F},\text{up}}$ and $K_{\mathcal{F},\text{down}}$ into (6.19):

$$\begin{aligned} K_{\mathcal{F},\text{now}} &= \hat{p}_{\text{up}} \left(\hat{p}_{\text{up}}\,(S + 2\,\epsilon) + \hat{p}_{\text{down}}\, S\right) + \hat{p}_{\text{down}} \left(\hat{p}_{\text{up}}\, S + \hat{p}_{\text{down}}\,(S - 2\,\epsilon)\right) \\ &= \hat{p}_{\text{up}}^2\,(S + 2\,\epsilon) + 2\,\hat{p}_{\text{up}}\,\hat{p}_{\text{down}}\, S + \hat{p}_{\text{down}}^2\,(S - 2\,\epsilon) \\ &= \frac{(1-\lambda)^2}{4}\,(S + 2\,\epsilon) + \frac{(1-\lambda)\,(1+\lambda)}{2}\, S + \frac{(1+\lambda)^2}{4}\,(S - 2\,\epsilon) \\ &= S - 2\,\epsilon\,\lambda. \end{aligned} \tag{6.21}$$

By working backwards this way, we have used the original pseudo path probabilities, in contrast to the forward price which uses the final-state probabilities. When the interest rate is not random, the two sets of probabilities are the same and so the forward price equals the futures price. The difference between the forward and futures prices is

$$K_{\mathcal{F},\text{now}} - K_F = \frac{\delta\,\epsilon\,(1-\lambda^2)}{1 + r_{1,\text{now}} + \delta\,\lambda},$$

the sign of which is determined by the covariance between the value of the underlier and the interest rate, $\delta\,\epsilon$.

Appendix A. Stochastic processes

Almost everything we encounter in asset pricing is a *stochastic process*. Quite simply, a stochastic process is something that evolves randomly through time according to a set of probabilistic rules, like the value of the stock. (Things that evolve deterministically, like the value of the bond, are special cases of stochastic processes.) For our purposes, we can think of as stochastic process as a description of "how something changes." For the stock and the bond, we specified (in addition to their values right now) each of their values in both states of the world. It often turns out to be convenient to specify instead the average change and the volatility of the changes.

For the purpose of exposition, let X_{now} be the value of an arbitrary stochastic process right now and let X_{up} and X_{down} be the values in the up and down states next year. For simplicity, let the probabilities that the stock price goes up or down be

$$p_{\text{up}} = \frac{1}{2} \quad \text{and} \quad p_{\text{down}} = \frac{1}{2}.$$

The change in D is a random variable that we will write as[10]

$$\Delta X = \begin{cases} X_{\text{up}} - X_{\text{now}} & \text{with probability } \frac{1}{2} \\ X_{\text{down}} - X_{\text{now}} & \text{with probability } \frac{1}{2}. \end{cases}$$

[10]The Δ in this appendix is not the same as the "deltas" we used in the section on options and futures.

We are particularly interested in (i) the average value of ΔX and (ii) the dispersion of ΔX. We call the average change the *expected change* of X:

$$\begin{aligned}\mu_X &= p_{\text{up}}\,(X_{\text{up}} - X_{\text{now}}) + p_{\text{down}}\,(X_{\text{down}} - X_{\text{now}})\\ &= \frac{X_{\text{up}} + X_{\text{down}}}{2} - X_{\text{now}}\\ &= X_{\text{avg}} - X_{\text{now}}.\end{aligned}$$

We will measure the dispersion by the *volatility* of the change:

$$\begin{aligned}\sigma_X &= \big((X_{\text{up}} - X_{\text{now}}) - (X_{\text{down}} - X_{\text{now}})\big)\sqrt{p_{\text{up}}\,p_{\text{down}}}\\ &= \frac{X_{\text{up}} - X_{\text{down}}}{2}\\ &= X_{\text{vol}}.\end{aligned}$$

We can see how the volatility, σ_X, is related to two other measures of the dispersion, (i) the variance and (ii) the standard deviation. The *variance* of the changes is given by the square of the volatility:

$$\begin{aligned}X_{\text{var}} &= p_{\text{up}}\big((X_{\text{up}} - X_{\text{now}}) - \mu_X\big)^2 + p_{\text{down}}\big((X_{\text{down}} - X_{\text{now}}) - \mu_X\big)^2\\ &= (X_{\text{up}} - X_{\text{down}})^2\,p_{\text{up}}\,p_{\text{down}}\\ &= \frac{(X_{\text{up}} - X_{\text{down}})^2}{4}\\ &= \sigma_X^2.\end{aligned}$$

The *standard deviation* is the square root of the variance, and hence the absolute value of the volatility:

$$X_{\text{dev}} = \sqrt{X_{\text{var}}} = \frac{|X_{\text{up}} - X_{\text{down}}|}{2} = |\sigma_X|.$$

With these definitions of the expected change and the volatility, we can write the change in the value of the stochastic process as

$$\Delta X = \begin{cases}\mu_X + \sigma_X & \text{with probability } \frac{1}{2}\\ \mu_X - \sigma_X & \text{with probability } \frac{1}{2}.\end{cases}$$

We can complete our description of the stochastic process by inventing a fundamental stochastic process, W:

$$\Delta W = \begin{cases}+1 & \text{with probability } \frac{1}{2}\\ -1 & \text{with probability } \frac{1}{2}.\end{cases}$$

Notice that ΔW has the following properties:

$$\mu_W = 0 \quad \text{and} \quad \sigma_W = 1.$$

We can write the random change in X in terms of the random change in W:

$$\Delta X = \mu_X + \sigma_X\,\Delta W.$$

For those cases where X_{now} is positive, we can look at the stochastic process for the *relative change* in X:

$$\frac{\Delta X}{X_{\text{now}}} = \mu_x + \sigma_x \, \Delta W,$$

where

$$\mu_x = \frac{\mu_X}{X_{\text{now}}} \quad \text{and} \quad \sigma_x = \frac{\sigma_X}{X_{\text{now}}}$$

are (respectively) the relative expected change and the relative volatility. (Note the use of the lower case letter.) If X is the value of an asset (and X_{now} is positive), then we can interpret $\Delta X / X_{\text{now}}$ as the random *return*, in which case μ_x is the expected return and σ_x is the volatility of the expected return.

If we have two stochastic processes, say X and Y, we may wish to the covariance between the changes in X and the changes in Y. It turns out that this covariance equals the product of the volatilities:

$$\begin{aligned}
\text{Cov}_{X,Y} = {} & \frac{1}{2} \big((X_{\text{up}} - X_{\text{now}}) - \mu_X\big) \big((Y_{\text{up}} - Y_{\text{now}}) - \mu_Y\big) \\
& + \frac{1}{2} \big((X_{\text{down}} - X_{\text{now}}) - \mu_X\big) \big((Y_{\text{down}} - Y_{\text{now}}) - \mu_Y\big) \\
= {} & \sigma_X \, \sigma_Y .
\end{aligned}$$

Risk and return. In the previous section, we found expressions for asset prices that involved averages calculated using pseudo probabilities. Using the pseudo probabilities, the expected change in the asset's value is given by

$$\begin{aligned}
\hat{\mu}_D &= \hat{p}_{\text{up}} \, (D_{\text{up}} - D_{\text{now}}) + \hat{p}_{\text{down}} \, (D_{\text{down}} - D_{\text{now}}) \\
&= (\hat{p}_{\text{up}} \, D_{\text{up}} + \hat{p}_{\text{down}} \, D_{\text{down}}) - D_{\text{now}} \\
&= \hat{D}_{\text{avg}} - D_{\text{now}}.
\end{aligned}$$

With this notation, we can write absence of arbitrage condition (4.6) as

$$\hat{\mu}_D = r \, D_{\text{now}}. \tag{A.1}$$

Let's examine the relationship between the expected change using pseudo probabilities and the expected change using the true probabilities:

$$\begin{aligned}
\mu_X - \hat{\mu}_X &= \left(\frac{1}{2} - \hat{p}_{\text{up}}\right) X_{\text{up}} + \left(\frac{1}{2} - \hat{p}_{\text{down}}\right) X_{\text{down}} \\
&= \left(\frac{1}{2} - \hat{p}_{\text{up}}\right) X_{\text{up}} + \left(\hat{p}_{\text{up}} - \frac{1}{2}\right) X_{\text{down}} \\
&= \left(\frac{1}{2} - \hat{p}_{\text{up}}\right) (X_{\text{up}} - X_{\text{down}}) \qquad \text{(A.2a)} \\
&= (1 - 2\,\hat{p}_{\text{up}}) \left(\frac{X_{\text{up}} - X_{\text{down}}}{2}\right) \\
&= \lambda \, \sigma_X,
\end{aligned}$$

where

$$\lambda = 1 - 2\,\hat{p}_{\rm up} \tag{A.2b}$$

is the *price of risk*. (We will give some meaning to this phrase below.) Notice that λ does not depend on $X_{\rm now}$, $X_{\rm up}$, or $X_{\rm down}$.

Using (A.2a), we can write (A.1) as

$$\mu_D = r\,D_{\rm now} + \lambda\,\sigma_D,$$

which, if $D_{\rm now} > 0$, we can write in terms of returns:

$$\mu_d = r + \lambda\,\sigma_d.$$

References

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