AN ANALYSIS OF THE DOUBLING STRATEGY: THE COUNTABLE CASE

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Preliminary and incomplete

Abstract. We analyze the doubling strategy in static and dynamic settings with a countable state space. We apply the no-arbitrage and no-free-lunch definitions of Kreps (1981), which (in the dynamic setting) put the focus on the gain produced by a self-financing trading strategy, rather than on the strategy itself. By applying the Krepsian notions of no arbitrage and no free lunches to dynamic models, instead of the notions common in standard practice, we avoid the situation where there are no free lunches at the same time there are arbitrage opportunities. Depending on the topological space one adopts, the doubling strategy is either (*i*) not in the space of payouts (and hence not a free lunch), (*ii*) in the space and a free lunch, or (*iii*) in the space but not a free lunch. In the latter case, which requires 'near risk-neutrality', the doubling strategy has a bubble component in the sense of Gilles and LeRoy (1997).

Introduction

In their seminal paper, Harrison and Kreps (1979, p. 400) refer to

> the well known doubling strategy by which one is *sure to win* at roulette: Bet on red, and keep doubling your bets until red comes out. To effect this strategy, you must be able to bet a countable number of times, although you will only bet a finite number of times in any particular state. [Emphasis added.]

This strategy is almost universally viewed as a pathological problem in dynamic security market models with infinite state spaces—a problem that requires side conditions to rule it out. Yet no special side conditions are required to rule out the doubling strategy in static models with infinite state spaces. We show that the static analysis can be applied to the dynamic setting in a natural way, thereby rendering the side conditions unnecessary.

The doubling strategy poses a problem because it costs nothing and but converges with probability one to something positive. The side conditions typically restrict the trading strategies in such a way as to remove the doubling strategy from the choice set. By contrast, we view the convergence of the doubling strategy as a

Date: April 3, 2002.

Fisher thanks Dan Waggoner for many useful conversations. The views expressed herein are the authors' and do not necessarily reflect those of the Federal Reserve Bank of Atlanta, the Federal Reserve System, or Bear, Stearns & Co.

topological issue. We see no compelling reason to adopt the topology of convergence in probability that underlies the convergence of the doubling strategy according to the standard analysis. With a stronger topology—such as the norm topology—fewer sequences converge and more functions are continuous. In many cases, the doubling strategy fails to converge in the norm topology.

A benefit of our approach is that the notions of arbitrage opportunities and free lunches that Kreps (1981) formalizes in the static setting can be directly applied to the dynamic setting. According the Kreps, 'no arbitrage' is a linear-algebraic concept, while 'no free lunches' is a topological concept. Moreover, there is a hierarchy: 'no free lunches' implies 'no arbitrage opportunities'. This contrasts with what has become standard practice in dynamic security market models: The linear-algebraic notion of 'no arbitrage' is missing. In its place, the dynamic notion of 'no arbitrage' is built on an implicitly topological foundation: A gain process generated by a self-financing trading strategy is allowed to converge in the topology of convergence in probability, regardless of the topological space on which preferences are defined. It should not be surprising that two distinct topologies would be at war with each other. Indeed, we provide an example in which, according to standard practice, there are no free lunches and yet there are arbitrage opportunities.

The doubling strategy can be viewed as the archetypical free-lunch sequence in the sense of Kreps (1981). Kreps emphasized the importance of fixing a space X of commodity bundles (security payouts) over which utility is defined, a cone K to determine positiveness, and a topology τ to determine continuity. Pricing operators, preferences, and free lunches can only be defined in terms of the triple (X, K, τ). Whether a sequence (or net, more generally) converges to a payout in chosen space depends crucially on the chosen topology. Whether an agent's preferences are continuous depends crucially on the chosen topology. By the same token, whether a linear functional (such as a price system) is continuous depends on the topology.

We embed the doubling strategy in a simple countable state space that we treat first in a static setting and then in a dynamic setting. In the static setting, we consider a number of topological spaces, including the space of charges. Depending on the topological space one adopts, the doubling strategy is either (*i*) not in the space of payouts (and hence not a free lunch), (*ii*) in the space and a free lunch, or (*iii*) in the space but not a free lunch. In the latter case, which requires what we call 'near risk-neutrality', the doubling strategy has a bubble component in the sense of Gilles and LeRoy (1997).[1] One of the examples we treat is a static version the example of Back and Pliska (1991).[2]

In the dynamic setting, we append a filtration to the probability space of the static setting and introduce stock-price dynamics adapted to the filtration. After characterizing the gain processes from self-financing trading strategies and introducing the state-price process, we identify the space of marketed securities with the

[1] Gilles and LeRoy (1997, Section 6.3) describe how their analysis applies in this case.

[2] Werner (1997) provides extensive treatment of the same static version of their example that we present here. In a related static model Gilles and LeRoy (1998) discuss many of the issues raised by Back and Pliska.

set of stopped gain processes. This identification allows us to directly apply Kreps' analysis in the dynamic setting, which is the primary contribution of the paper. As a consequence, we are able to identify (i) the absence-of-arbitrage conditions with the existence of a state-price process (we include strict positivity by definition) and (ii) the existence of an equivalent martingale measure with the existence of a uniformly integrable state-price process. Moreover, we show that if the state-price process has limit points in the topological dual space, then there are no free lunches.

Arbitrage and free lunches are about getting something for nothing. As such they are about the gains that are produced by self-financing trading strategies rather than the trading strategies themselves. Our approach puts the focus on the gain process rather than on the trading strategy that generates the gain. In addition, our topological approach enables us to accommodate the Gilles and LeRoy (1997) model of rational bubbles; we incorporate their analysis of the doubling strategy into an explicitly dynamic model.

To highlight the differences between our approach and the standard approach of analyzing dynamic security market models, we revisit the example of Back and Pliska (1991) in their dynamic setting. We keep the probability space and the underlying state prices that are generated by their stock-price dynamics. We then modify the stock-price dynamics and apply the analysis of Kreps. We find there are no free lunches (in L^∞ equipped with the sup norm topology), and *a fortiori* there are no arbitrage opportunities. This is not surprising since we have changed neither the measure nor the state prices. However, if we were to adopt the standard definition of no arbitrage (as Back and Pliska do), the very same economy must be said to have arbitrage opportunities, even though there are no free lunches. The so-called arbitrage opportunities appear because the trading strategy that implements the doubling strategy is fundamentally bounded: The number of shares of the stock that are held is bounded, the amount borrowed is bounded, and the gain is bounded. This allows the doubling strategy to be an arbitrage by their definition, even though it is not a free lunch because it does not converge in the topological space.

Although we restrict ourselves to the countable case in this paper, our approach applies more generally. As part of our ongoing research, we are in the process of extending our approach to the continuous-time Black–Scholes model and to the general class of semi-martingales.

Outline of paper. In Section 1, we provide a quick summary of Kreps (1981). In Section 2 we present the model in its static setting, and in Section 3 we present the model in its dynamic setting.

1. Kreps: Arbitrage and free lunches

In this section, we provide a quick summary of Kreps (1981), focusing narrowly on what is important for our analysis.

In what follows, X is a real linear space, K is a cone in X with the origin deleted, τ is a linear topology on X. Ψ is the set of τ continuous and K strictly positive linear functionals on X. M is a subspace of X that represents marketed bundles of goods that can be constructed from a set of marketed securities $M_0 \subseteq X$. (We will typically assume there are an infinite number of marketed securities.) In particular, M is the span of M_0, such that $m \in M$ if $m = \sum_i \lambda_i m_i$ for $m_i \in M_0$ where there are a finite number of non-zero λ_i. π is a linear functional on $M \subseteq X$ (representing the market prices of payoffs in M) constructed from a function $\pi_0 : M_0 \to \mathbb{R}$ as follows: $\pi(m) = \sum_i \lambda_i \pi_0(m_i)$ for $m = \sum_i \lambda_i m_i \in M$.

Kreps defines what he calls a *free lunch* in an infinite-dimensional space: A free lunch is a net $\{(m_\alpha, x_\alpha)\} \subseteq M \times X$ and a bundle $k \in K$ such that[3]

$$(i)\ m_\alpha - x_\alpha \in K \cup \{0\} \text{ for all } \alpha,\ (ii)\ x_\alpha \xrightarrow{\tau} k, \text{ and } (iii) \lim_\alpha\ \pi(m_\alpha) \le 0. \tag{1.1}$$

We cannot emphasize too strongly that the convergence of x_α to k in (1.1) depends on the topology τ. The absence of arbitrage is defined as follows: The pair (M, π) admits no arbitrage opportunities if

$$m \in M \cap K \implies \pi(m) > 0. \tag{1.2}$$

The absence of arbitrage is necessary for the absence of free lunches, but not sufficient in general. Note that the absence of arbitrage pertains only to the marketed subspace M, while the absence of free lunches pertains its topological closure $[M]$.

Kreps defines the *viability* of preferences. An agent is specified by a complete and transitive binary relation $\succsim$ on X representing the agent's preferences for net trades. Preferences are *convex*, τ *continuous*, and K *strictly increasing* if they satisfy (respectively) the following three conditions:

$$x, x' \succsim x'' \text{ and } \lambda \in [0,1] \text{ imply } \lambda x + (1-\lambda) x' \succsim x'' \tag{1.3a}$$

$$\text{for all } x \in X, \text{ the sets } \{x' \in X : x \succsim x'\} \text{ and } \{x' \in X : x' \succsim x\} \text{ are closed in } \tau \tag{1.3b}$$

$$\text{for all } x \in X \text{ and } k \in K,\ x + k \succ x. \tag{1.3c}$$

The pair (M, π) is said to be viable if there is some convex, τ continuous, and strictly increasing preference relation $\succsim$ on X and some $m^* \in M$ such that

$$\pi(m^*) \le 0 \text{ and } m^* \succsim m \text{ for all } m \in M \text{ such that } \pi(m) \le 0. \tag{1.4}$$

Here are two of the main theorems that Kreps presents. Take as given the space–cone–topology triple (X, K, τ).

Theorem 1.1 (Extension)**.** *The pair (M, π) is viable if and only if there exists $\psi \in \Psi$ such that ψ extends π to all of X.*

Theorem 1.2 (No free lunches)**.** *If the pair (M, π) is viable, then the pair admits no free lunches.*

[3]Nets are generalizations of sequences. In some topological spaces, sequences are insufficient to characterize convergence. Clark (1993) provides a stronger notion of a free lunch (for which $\pi(m_\alpha) \le 0$) that is for all intents and purposes equivalent.

2. Static setting

Here we present the ideas in this paper in a simple static setting where we apply the Krepsian analysis.[4] We refer the reader to Appendix A for omitted details.[5]

Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ be the set of natural numbers. All the spaces we consider are subspaces of $\mathbb{R}^{\mathbb{N}}$, the vector space of all real sequences on $\mathbb{N}$. Take as given the measure space $(\mathbb{N}, 2^{\mathbb{N}}, P)$, where $2^{\mathbb{N}}$ is the set of all subsets of $\mathbb{N}$, and P is a probability measure characterized by $P_\omega := P[\{\omega\}] > 0$ and $\sum_{\omega=1}^{\infty} P_\omega = 1$. For example, $P_\omega = \rho\,(1-\rho)^{\omega-1}$, where $0 < \rho < 1$.

The set of marketed claims M_0 is comprised of a complete set of Arrow–Debreu securities $\delta_i(\omega) = \mathbf{1}_{\{i\}}(\omega)$ and a 'bond' that pays one unit in every state, $B(\omega) = 1$. The given prices of the marketed securities are $\pi_0(\delta_i) = \mathit{\Gamma}_i$ and $\pi_0(B) = 1$. By definition, $m \in M$ if and only if $m = \alpha_0\, B + \sum_{i=1}^{i^*} \alpha_i\, \delta_i$, for some $i^* \in \mathbb{Z}_+ = \{0, 1, 2, \cdots\}$. The state-by-state payoffs for $m \in M$ are

$$m(\omega) = \begin{cases} \alpha_0 + \alpha_\omega & \omega \le i^* \\ \alpha_0 & \omega > i^*. \end{cases} \tag{2.1}$$

Define the pricing functional $\pi : M \to \mathbb{R}$ as

$$\begin{aligned} \pi(m) &:= \alpha_0\, \pi_0(B) + \sum_{i=1}^{i^*} \alpha_i\, \pi_0(\delta_i) \\ &= \alpha_0 + \sum_{i=1}^{i^*} \alpha_i\, \mathit{\Gamma}_i \\ &= \sum_{i=1}^{i^*} (\alpha_0 + \alpha_i)\, \mathit{\Gamma}_i + \alpha_0 \left(1 - \sum_{i=1}^{i^*} \mathit{\Gamma}_i\right). \end{aligned} \tag{2.2}$$

The last line of (2.2) expresses $\pi(m)$ in terms of the prices of the first i^* Arrow–Debreu securities and the price of the portfolio's 'tail'. It is convenient to reexpress the pricing functional:

$$\pi(m) = E^P[m\, y] + \alpha_0 \left(1 - E^P[y]\right), \tag{2.3}$$

where

$$\begin{aligned} y(\omega) &:= \mathit{\Gamma}_\omega / P_\omega \\ E^P[m\, y] &= \sum_{\omega=1}^{\infty} m(\omega)\, y(\omega)\, P_\omega = \sum_{\omega=1}^{\infty} m(\omega)\, \mathit{\Gamma}_\omega \\ E^P[y] &= \sum_{\omega=1}^{\infty} y(\omega)\, P_\omega = \sum_{\omega=1}^{\infty} \mathit{\Gamma}_\omega. \end{aligned}$$

[4] Werner (1997) analyzes a static version of the example in Back and Pliska (1991).

[5] Appendix A is incomplete.

Absence of arbitrage. We now address the absence of arbitrage opportunities. For $m \in M \cap K$, we must have $m(\omega) \geq 0$ for all $\omega \in \mathbb{N}$, with at least one strict inequality. If every Arrow–Debreu security and every tail has a positive price, then there are no arbitrage opportunities. Specifically, if

$$\varGamma_i > 0 \text{ for all } i \in \mathbb{N} \quad \text{and} \quad \sum_{i=1}^{\infty} \varGamma_i \leq 1, \tag{2.4}$$

then $m \in M \cap K \implies \pi(m) > 0$. We can reexpress (2.4) in terms of y:

$$y(\omega) > 0 \text{ for all } \omega \in \mathbb{N} \quad \text{and} \quad E^P[y] \leq 1. \tag{2.5}$$

No free lunches. Before we address free lunches per se, we consider the closure of M in the topological space (X, τ). Let $\ell^p(P) = L^p(\mathbb{N}, 2^{\mathbb{N}}, P)$ denote the space of sequences for which $x \in \ell^p(P)$ if $\|x\|_p^P < \infty$, and let τ_p^P denote the topology of convergence in the $\ell^p(P)$ norm.

Closure of M in (X, τ). Let $(X, \tau) = (\ell^p(P), \tau_p^P)$. Note that $M \neq X$ even though there is a complete set of Arrow–Debreu securities in M. Let $[M]_\tau$ denote the topological closure of M in τ. We now show that $[M]_\tau = X = \ell^p(P)$ for $p < \infty$. Every element $x \in X$ can be expressed in terms of Arrow–Debreu securities: $x = \sum_{i=1}^{\infty} \alpha_i \, \delta_i$, where $\|x\|_p^P = (\sum_{\omega=1}^{\infty} |\alpha_\omega|^p \, P_\omega)^{1/p} < \infty$. Define $x_j := \sum_{i=1}^{j} \alpha_i \, \delta_i$. For $p < \infty$, $\lim_{j \to \infty} \|x - x_j\|_p^P = 0$, confirming $x_j \xrightarrow{\tau_p^P} x$. The situation is different for $\ell^\infty(P)$. For example, for $\{1, 1, 1, \cdots\}$, the sup norm does not decrease in the tail. Without access to the bond,

$$[M]_\tau = \{x \in \mathbb{R}^{\mathbb{N}} : \lim_{i \to \infty} x_i = 0\}.$$

With the bond, the closure is larger, but still not the entire space:

$$[M]_\tau = \{x \in \mathbb{R}^{\mathbb{N}} : x_\infty = \lim_{i \to \infty} x_i \text{ exists in } \mathbb{R}\}.$$

Finally, let $(X, \tau) = (\ell^1(P)^{**}, w^*)$. The closure of M in $\ell^1(P)$ with the weak topology is $\ell^1(P)$ as before. Since $\ell^1(P)$ is dense in $\ell^1(P)^{**}$, the weak* closure is $[M]_\tau = \ell^1(P)^{**}$. The use of nets, however, is required to achieve the closure.

No free lunches. As noted in Section 1, $\varPsi \subset X^*$ is the set of strictly positive continuous linear functionals on (X, τ). Theorems 1.1 and 1.2 imply the following: If π extends to some $\psi \in \varPsi$, where $\psi(m) = \pi(m)$ for all $m \in M$, then there are no free lunches. We investigate the possibility of extending π in various topological spaces. In all cases, let $K = X_+ \setminus \{0\}$.

First let $(X, \tau) = (\ell^p(P), \tau_m^P)$, where τ_m^P is the topology of convergence in measure. As shown in Appendix A, $\varPsi = \varnothing$ in this case. Therefore, a strictly positive π cannot be extended and there exist free lunches.[6]

[6]Kreps showed in an example that it is not be possible to generate any free lunches if the market is sufficiently incomplete. However, the market here is sufficiently complete to ensure the existence of free lunches absent a suitable extension of π.

Next let $(X,\tau) = (\ell^p(P), \tau_p^P)$ where $1 \leq p < \infty$, in which case $X^* = \ell^q(P)$, where $q = p/(p-1)$. Every continuous linear functional ψ on (X,τ) has a Riesz Representation

$$\psi(x) = \sum_{\omega=1}^{\infty} x(\omega)\, y(\omega)\, P_\omega = E^P[x\, y], \tag{2.6}$$

where $y \in X^*$. Comparing (2.3) with (2.6), we must have $E^P[y] = 1$. Assuming this holds, the condition for π to extend to ψ reduces to

$$\left(\|y\|_q^P\right)^q = \sum_{\omega=1}^{\infty} y(\omega)^q\, P_\omega = \sum_{\omega=1}^{\infty} \Gamma_\omega^q\, P_\omega^{1-q} = \sum_{\omega=1}^{\infty} \Gamma_\omega^{p/(p-1)}\, P_\omega^{1/(1-p)} < \infty \tag{2.7}$$

for $p > 1$. For $p = 1$, the condition is that y be bounded. On the other hand, if $E^P[y] < 1$, then $\psi(B) \neq \pi_0(B) = 1$. In this case, π cannot be extended.

Now let $(X,\tau) = (\ell^\infty(P), \tau_M)$, where τ_M is the Mackey topology; i.e., the strongest topology for which $X^* = \ell^1(P)$. Since $y \in \ell^1(P)$ by construction, we only need to check $\psi(m) = \pi(m)$ for all $m \in M$. In this case, the representation for $\psi(x)$ is given by (2.6); hence, extension requires $E^P[y] = 1$.

Now let $(X,\tau) = (\ell^\infty(P), \tau_\infty^P)$. The dual space is $X^* = \ell^\infty(P)^* = \ell^1(P)^{**}$, where $\ell^1(P) \subset \ell^1(P)^{**}$. In this case, the Krein–Rutman Theorem guarantees the extension property in L^∞ of the pair (M,π) whenever the absence of arbitrage is satisfied (by virtue of the non-empty interior of L_+^∞).[7] There are two cases to consider. First, suppose $E^P[y] = 1$. In this case, the extension of π has the representation given in (2.6), which indeed agrees with π on M. Second, suppose $E^P[y] < 1$. In this case, the extension of π does not have such a representation. A more general representation involving finitely-additive set functions does exist. For $x \in \ell^\infty(P)$, we have

$$\psi(x) = E^P[x\, y] + \int_\Omega x\, d\Phi,$$

where $\Phi \in \boldsymbol{pa}$ is a pure charge such that $\int_\Omega m\, d\Phi = \alpha_0\,(1 - E^P[y])$ for all $m \in M$. It should be noted that Φ is not uniquely identified, which is not surprising given the incompleteness of the market in $(\ell^\infty(P), \tau_\infty^P)$.

Finally, let $(X,\tau) = (\ell^1(P)^{**}, w^*)$, the dual of which is $\ell^\infty(P)$. If $y \in \ell^\infty(P)$, then π extends to ψ, where for $x \in \ell^1(P)^{**}$, we have

$$\psi(x) = E^P[x^1\, y] + \int_\Omega y\, d\varphi,$$

where $x = x^1 + x^\perp$, with $x^1 \in \ell^1(P)$, $x^\perp \in \ell^1(P)^\perp$, and $\varphi \in \boldsymbol{pa}$ is the pure charge that corresponds to $x^\perp$.

[7]On may think of the Krein–Rutman Theorem as the 'flip side' of the Hahn–Banach Theorem. Here is a statement of the theorem in its geometric form [see Holmes (1975)]:

Theorem 2.1 (Krein–Rutman). *Let X be an ordered linear space, M be a subspace of X, K be the positive cone, and $M \cap K$ have an interior point of K. Then any positive linear functional on M admits an extension as a positive linear functional on X.*

Bubbles à la *Gilles and LeRoy.* Gilles and LeRoy (1992, 1997) present two conceptually different models of bubbles, both of which appeal to the theory of charges for the representation of values. Gilles and LeRoy (1992) model bubbles in the price system, while Gilles and LeRoy (1997) model payout bubbles.

First, we address bubbles in the price system. Let $(X, \tau) = (\ell^\infty(P), \tau^P_\infty)$ and assume $E^P[y] < 1$. Given $x \in X$, Gilles and LeRoy (1992) define the fundamental component as $E^P[x\,y]$ and the bubble component as $\psi(x) - E^P[x\,y] = \int_\Omega x\,d\Phi$. Given $x = \sum_{j=1}^\infty \alpha_j\,\delta_j$, define $x_i := \sum_{j=1}^\infty \alpha_j\,\delta_j$. Note $x_i \xrightarrow{P} x$. If $x_i \xrightarrow{\tau} x$, then the bubble value is zero. The bubble in the price system is reflected in the value of the bond: The fundamental value is $E^P[y]$ and the bubble value is $1 - E^P[y] = \int_\Omega d\Phi$. In this case, the value of the probability limit is greater than the limit of the values.

Now we address payout bubbles. Let $(X, \tau) = (\ell^1(P)^{**}, w^*)$ and assume $y \in \ell^\infty(P)$ so that π extends to ψ. Given $x \in X$, where $x = x^1 + x^\perp$, Gilles and LeRoy (1997) define the fundamental value as $E^P[x^1\,y]$ and the bubble component as $\psi(x) - E^P[x^1\,y] = \int_\Omega y\,d\varphi$. Consider $\{x_i\}_{i=1}^\infty$, where $x_i \in \ell^1(P)$ for all $i \in \mathbb{N}$, $\sup_{i\in\mathbb{N}}\ \|x_i\|^P_1 < \infty$, $x_i \xrightarrow{P} x^1 \in \ell^1(P)$, and $v = \lim_{i\to\infty}\ \psi(x_i)$ exists. Alaoglu's Theorem guarantees the existence of limit points in $\ell^1(P)^{**}$, all of which have value v. Then the fundamental value of the limit points is $E^P[x^1\,y]$ and the bubble value is $v - E^P[x^1\,y]$. For example, let $y \equiv 1$ and $x_i = \Gamma_i^{-1}\,\delta_i$, so that $\psi(x_i) = 1$ and $\|x_i\|^P_1 = 1$ for all $i \in \mathbb{N}$, and $x_i \xrightarrow{P} 0$. Consequently, the fundamental value is 0 and the bubble value is 1. In this case, the value of the probablity limit is less than the limit of the values.

Equivalent measure. Consider a topological space $(X, \tau) = (\ell^p(P), \tau^P_p)$ and a marketed subspace $M \subseteq X$. Assume $y(\omega) > 0$ for all $\omega \in \mathbb{N}$ and $E^P[y] = 1$. These conditions are sufficient to define an equivalent measure Q, where $Q[\{\omega\}] = y(\omega)\,P[\{\omega\}]$. The Radon–Nikodym derivative is $dQ/dP = y \in \ell^1(P)$ and $dP/dQ = 1/y \in \ell^1(Q)$. Whenever $E^P[|x\,y|] < \infty$, we have $E^P[x\,y] = E^Q[x]$.

We now show that if an equivalent measure Q exists, π extends to $\psi \in \Psi$ for $(\hat{X}, \hat{\tau}) = (\ell^1(Q), \tau^Q_1)$. First, note $M \subseteq \hat{X}$. Let $\psi(x) = E^Q[x]$ for all $x \in \hat{X}$, so that $\psi(m) = E^Q[m] = E^P[m\,y] = \pi(m)$ for all $m \in M$. This does not require $y \in X^*$, and we cannot conclude $E^P[x\,y]$ exists for all $x \in X$ unless $X = \ell^\infty(P)$. Therefore, there are no free lunches in $(\hat{X}, \hat{\tau})$ regardless of whether there are free lunches in (X, τ).

We now show that $\hat{X} \not\subseteq X$. Given $dQ/dP = y \in \ell^1(P)$, if $x \in X$ and $y \in X^*$, then $x\,y \in \ell^1(P)$ and $x \in \ell^1(Q)$. We also have $dP/dQ = 1/y \in \ell^1(Q)$. However, $1/y \notin X$ without further conditions. Under what conditions is $\|1/y\|^P_p$ finite?

$$\|1/y\|^P_p = \left(\sum_{\omega=1}^\infty \Gamma_\omega^{-p}\,P_\omega^{1+p}\right)^{1/p}.$$

For $P_\omega = \rho\,(1-\rho)^{\omega-1}$ and $\Gamma_\omega = \zeta\,(1-\zeta)^{\omega-1}$, we have $\|1/y\|^P_p < \infty$ if $\rho = \zeta$ or $\rho > 1-(1-\zeta)^{p/(1+p)}$. Compare this with the no-free lunch condition $\|y\|^P_{p/(p-1)} < \infty$, which is $\rho = \zeta$ or $\rho < 1 - (1 - \zeta)^p$. For $\zeta = 1/2$ and $p = 1$, we have $1 - (1/2)^{1/2} < \rho \leq 1/2$.

Preferences. We want to emphasize that the choice of the topological space of payouts (X, τ) is a choice about the continuity of preferences. Kreps' Extension Theorem 1.1 says that if π cannot be extended, there are no continuous preferences (that are also strictly increasing and convex) that can support the price system. Thus, the adoption of the topological space $(\ell^p(P), \tau_m^P)$, for which extension can never occur (owing to $\Psi = \varnothing$), rules out such continuous preferences. For example, consider the following 'risk-neutral' utility function: $U(x) = E^P[x] = \sum_{\omega=1}^{\infty} x(\omega)\, P_\omega$, which is linear and convex. This utility function is not continuous on $(\ell^p(P), \tau_m^P)$.

The doubling strategy. Here is an informal statement of the classic doubling strategy. Make a bet that pays 1 if 'red' occurs on the first spin of the wheel and pays -1 otherwise. Stop if 'red' occurs (and keep 1); otherwise borrow 1 to pay the loss and double the bet to 2 for the next spin. Stop if 'red' occurs (pay off the debt and keep 1); otherwise borrow 3 to payoff the loss and the accumulated debt and redouble the bet to 4. Continue this pattern, but stop after the n-th spin regardless of the outcome. The doubling strategy refers to the limiting case as n goes to infinity; in other words, continue until 'red' first occurs. This limiting strategy converges in measure to the number 1.

Now we turn to a formal statement. Assume there are no arbitrage opportunities. The doubling strategy is a sequence of portfolios $\{z_i\}_{i=1}^{\infty} \subset M$ with the following two features: $\pi(z_i) = 0$ for all $i \in \mathbb{N}$ and $z_i \xrightarrow{P} 1$. The i-th portfolio is given by

$$z_i := (1 - \beta_i)\, B + \beta_i \sum_{j=1}^{i} \delta_j, \quad \text{where } \beta_i := \frac{1}{1 - \sum_{j=1}^{i} \varGamma_j}. \tag{2.8}$$

The state-by-state payouts to z_i are

$$z_i(\omega) = \begin{cases} 1 & \omega \leq i \\ 1 - \beta_i & \omega > i. \end{cases}$$

The doubling-strategy sequence $\{z_i\}$ converges almost surely to 1:

$$\lim_{i \to \infty} z_i(\omega) = 1 \quad \text{for all } \omega \in \mathbb{N}.$$

Almost sure convergence implies convergence in measure.

In the classic doubling strategy (described above), $\varGamma_i = 2^{-i}$ and $\beta_i = 2^i$, so that each time 'red' fails to appear, there is a literal doubling. In general, however, the sequence $\{\beta_i\}$ does not literally involve doubling. It is increasing, with $\beta_1 > 1$. It is unbounded if $\sum_{\omega=1}^{\infty} \varGamma_\omega = 1$; otherwise, it is bounded.

If the pricing functional π extends to $\psi \in \Psi$, then there are no free lunches, and consequently the doubling strategy has no limit points in K. We now consider a number of topological spaces to examine the conditions under which the doubling strategy does in fact converge or have limit points in K. First, the doubling strategy is always a free lunch in $(\ell^p(P), \tau_m^P)$, because $\Psi = \varnothing$. Next, let us examine the convergence of $\{z_i\}$ in $(X,\tau) = (\ell^p(P), \tau_p^P)$. In particular,

$$\|1 - z_i\|_p^P = \beta_i \left(\sum_{\omega=i+1}^{\infty} P_\omega \right)^{1/p} = \frac{\left(1 - \sum_{\omega=1}^{i} P_\omega\right)^{1/p}}{1 - \sum_{\omega=1}^{i} \Gamma_\omega}. \tag{2.9}$$

Clearly, if $\sum_{\omega=1}^{\infty} \Gamma_\omega < 1$, then $\lim_{i\to\infty} \|1 - z_i\|_p^P = 0$ for $1 \le p < \infty$, but not for $p = \infty$. On the other hand, if $\sum_{\omega=1}^{\infty} \Gamma_\omega = 1$, the convergence of $\{z_i\}$ depends on the limiting properties of P_ω and Γ_ω.

Under what circumstances is the doubling strategy not a free lunch and yet is in $(X,\tau) = (\ell^1(P)^{**}, w^*)$? We assume $\sum_{\omega=1}^{\infty} \Gamma_\omega = 1$, for otherwise the doubling strategy would be a free lunch in $\ell^1(P) \subset \ell^1(P)^{**}$. In addition, we require $y \in \ell^\infty(P)$; i.e., the boundedness of y. If $\{z_i\}$ is bounded in $\ell^1(P)$, then it will have limit points in $\ell^1(P)^{**}$:

$$\|z_i\|_1^P = \sum_{\omega=1}^{i} P_\omega + (\beta_i - 1) \sum_{\omega=i+1}^{\infty} P_\omega = 1 - 2 \left(\sum_{\omega=i+1}^{\infty} P_\omega \right) + \frac{\sum_{\omega=i+1}^{\infty} P_\omega}{\sum_{\omega=i+1}^{\infty} \Gamma_\omega}.$$

We see that the boundedness of $\{z_i\}$ in $\ell^1(P)$ requires $1/y \in \ell^\infty(P)$.[8] We refer to the case where both y and $1/y$ are bounded as 'near risk neutrality.' The risk neutral case is characterized by $y \equiv 1$.[9]

The suicide strategy. The suicide strategy is closely-related to the doubling strategy: $\{Z_i\}_{i=1}^{\infty}$, where $Z_i = 1 - z_i$ [see (2.8)]. Note that $Z_i \xrightarrow{P} 0$, $\pi(Z_i) = 1$ for all $i \in \mathbb{N}$, and $\|Z_i\|_p^P = \|1 - z_i\|_p^P$ [see (2.9)]. If $\{Z_i\}$ is bounded in $\ell^1(P)$, then it has pure charge limit points. If it converges in $\ell^1(P)$, then it converges to zero.

Two examples. Here we provide two arbitrage-free examples that we will use throughout the paper to investigate free lunches in general and the doubling strategy in particular. In both examples, we adopt the probability measure $P_\omega = \rho\,(1-\rho)^{\omega-1}$, where $0 < \rho < 1$.

Example 1. In this example, $\Gamma_\omega = \zeta\,(1-\zeta)^{\omega-1}$, where $0 < \zeta < 1$, so that $\sum_{\omega=1}^{\infty} \Gamma_\omega = 1$ and $y(\omega) = \Gamma_\omega / P_\omega = (\zeta/\rho)\,\big((1-\zeta)/(1-\rho)\big)^{\omega-1}$. There are no arbitrage opportunities and an equivalent measure exists: $Q_\omega = \Gamma_\omega$. The doubling strategy sequence is $\beta_i = (1-\zeta)^{-i}$. The classic doubling strategy is given by $\rho = \zeta = 1/2$. If $\rho \le \zeta$, then y is bounded and $y \in \ell^\infty(P)$. The $\ell^q(P)$ norm is given by

$$\|y\|_q^P = \begin{cases} 1 & \rho = \zeta \\ \left(\frac{\zeta^q\,(1-\rho)^q\,\rho^{1-q}}{(1-\rho)^q - (1-\rho)\,(1-\zeta)^q} \right)^{1/q} & \rho < 1 - (1-\zeta)^{q/(q-1)} \\ \infty & \text{otherwise.} \end{cases}$$

[8] If $1/y$ is bounded, then no pure charges have zero value (other than the zero charge). If $y_i \to 0$, then all pure charges have zero value. Otherwise, ($1/y$ is not bounded and $y_i \not\to 0$) some pure charges have zero value and others do not.

[9] Risk neutrality is sufficient for $y \equiv 1$, but not necessary.

Given $q = p/(p-1)$, the condition $\|y\|_q^P < \infty$ is equivalent to $\rho = \zeta$ or $\rho < 1-(1-\zeta)^p$.

We consider two topological spaces. First, let $(X, \tau) = (\ell^p(P), \tau_p^P)$, so that $X^* = \ell^{p/(p-1)}(P)$. If $\rho < 1-(1-\zeta)^p$ or $\rho = \zeta$, then $y \in X^*$, in which case there are no free lunches. For the doubling strategy, we have

$$\lim_{i\to\infty} \|1 - z_i\|_p^P = \lim_{i\to\infty} \left(\frac{(1-\rho)^{1/p}}{1-\zeta}\right)^i = \begin{cases} \infty & \rho < 1-(1-\zeta)^p \\ 1 & \rho = 1-(1-\zeta)^p \\ 0 & \rho > 1-(1-\zeta)^p. \end{cases}$$

Clearly, the doubling strategy does not converge when π extends. There is one case where the doubling strategy does not converge even though π does not extend: $\rho = 1-(1-\zeta)^p$ for $p > 1$. Nevertheless, there are free lunches in this case,[10] but the doubling strategy is not one of them.

Second, let $(X, \tau) = (\ell^1(P)^{**}, w^*)$. In this case, $X^* = \ell^\infty(P)$. Therefore, there are no free lunches if $\rho \leq \zeta$. There will be limit points in X if the doubling-strategy gain is bounded in $\ell^1(P)$. We have

$$\lim_{i\to\infty} \|z_i\|_1^P = \lim_{i\to\infty} 1 - 2\,(1-\rho)^i + \left(\frac{1-\rho}{1-\zeta}\right)^i = \begin{cases} \infty & \rho < \zeta \\ 2 & \rho = \zeta \\ 1 & \rho > \zeta, \end{cases}$$

which shows that $\{z_i\}$ is bounded for $\rho \geq \zeta$. For $\rho < \zeta$, there are no free lunches, but $\{z_i\}$ is not bounded in $\ell^1(P)$, and there are no limit points. For $\rho > \zeta$, there are free lunches, and $\{z_i\}$ is bounded in $\ell^1(P)$, and therefore it has free-lunch limit points in X. For $\rho = \zeta$, there are no free lunches, and the sequence $\{z_i\}$ is bounded in $\ell^1(P)$. Consequently, Alaoglu's Theorem guarantees limit points in $\ell^1(P)^{**}$. As Fisher and Gilles show, the $\ell^1(P)$ component of every limit point is B. The $\ell^1(P)^\perp$ component is not uniquely identified by the sequence $\{z_i\}$. However, if $\{G_\alpha\}_{\alpha\in A}$ is a convergent subnet of $\{z_i\}$, then $G_\alpha \xrightarrow{w^*} \mathcal{G} \in \ell^1(P)^{**}$, where $\mathcal{G} = B + \mathcal{G}_*$, $B \in \ell^1(P)$, and $\mathcal{G}_* \in \ell^1(P)^\perp$; note that $B > 0$, but $\mathcal{G}_* < 0$, so $\mathcal{G} \notin K$. In conclusion, the only case for which the doubling strategy is in the space and not a free lunch is the 'risk-neutral' case $\rho = \zeta$.

Note the following about the limit of the $\ell^1(P)$ norm of the suicide strategy and the limit of $1/y$:

$$\lim_{i\to\infty} \|Z_i\|_1^P = \lim_{\omega\to\infty} \frac{1}{y(\omega)} = \begin{cases} \infty & \rho < \zeta \\ 1 & \rho = \zeta \\ 0 & \rho > \zeta. \end{cases}$$

For $\rho > \zeta$, the suicide strategy converges in the $\ell^1(P)$ norm to zero (since y is not bounded). For $\rho < \zeta$, $y \to 0$ and all pure charges have zero value. [So why does the norm explode?]

[10] We need to show one.

Example 2. This example is related to Back and Pliska (1991) and Werner (1997).[11] In this example, $\mathit{\Gamma}_\omega = \big((2\,\omega)\,(1+\omega)\big)^{-1} > 0$, where $\sum_{\omega=1}^{\infty} \mathit{\Gamma}_\omega = 1/2$. Let $(X,\tau) = (\ell^p(P), \tau_p^P)$. For $1 \le p < \infty$, π cannot be extended and there are free lunches. For $p = \infty$, the Krein–Rutman Theorem guarantees π can be extended and consequently there are no free lunches. The doubling strategy sequence is $\beta_i = 2\,(1+i)/(2+i)$. We have

$$\lim_{i\to\infty} \|1 - z_i\|_p^P = \lim_{i\to\infty} \frac{2\,(1+i)\,(1-\rho)^{i/p}}{2+i} = \begin{cases} 0 & 1 \le p < \infty \\ 2 & p = \infty, \end{cases}$$

which shows that the doubling strategy converges if and only if π fails to extend.

3. Dynamic setting

We adopt the measure space from the preceeding section, $(\mathbb{N}, 2^{\mathbb{N}}, P)$, to which we add a filtration. The information structure is given as follows. Define the following partition of the state space into $i+1$ sets:

$$R_i := \{\{1\},\{2\},\cdots,\{i\},\{i+1,i+2,\cdots\}\}. \tag{3.1}$$

Let $\mathcal{F}_i$ be the set consisting of all unions of sets from R_i. Then $\{\mathcal{F}_i\}_{i=0}^{\infty}$ is a sequence of increasing σ-algebras that we take as the filtration, where $\mathcal{F}_0 = \{\varnothing, \mathbb{N}\}$ and $\lim_{i\to\infty} \mathcal{F}_i = \mathcal{F} = 2^{\mathbb{N}}$.

We interpret P_ω as the probability that 'red' first occurs on spin ω of (possibly biased) roulette wheel. The conditional probability that 'red' occurs on the next spin given that it has not yet occured is given by

$$q_{i+1} := P[\{i+1\}|\omega > i] = \frac{P_{i+1}}{\sum_{\omega=i+1}^{\infty} P_\omega}.$$

Obviously, $P[\{\omega : \omega > i+1\}|\omega > i] = 1 - q_{i+1}$. These conditional probabilities will be used in computing conditional expectations. The geometric distribution is a simple example: $P_\omega = \rho\,(1-\rho)^{\omega-1}$ and $q_i = \rho$.

Define $E_i^P[x] := E^P[x|\mathcal{F}_i]$ and note that $E_0^P[x] = E^P[x]$. In a discrete-index setting (such as we have here), a process X is a martingale [relative to $(\{\mathcal{F}_i\}, P)$] if X is adapted to $\{\mathcal{F}_i\}$, $X_i \in \ell^1(P)$ for all $i \ge 0$, and $E[X_{i+1}|\mathcal{F}_i] = X_i$ for all $i \ge 0$.[12]

Consider a market for trading two securities at a countable number of times $0 = t_0 < t_1 < \cdots < T$.[13] An example of the topological space of payouts at time T is (X,τ) where $X = \ell^p(P) = \ell^p(\mathbb{N}, 2^{\mathbb{N}}, P)$ for $1 \le p \le \infty$ and τ is τ_p^P, the $\ell^p(P)$ norm topology.[14] The price of one security (the bond or 'money-market account') at time t_i equals 1 in every state: $B_i(\omega) = 1$. The state-by-state price of the second security (the 'stock') at time t_i is given by

$$S_i(\omega) = \begin{cases} f_\omega & \omega \le i \\ g_i & \omega > i, \end{cases} \tag{3.2}$$

[11] See also Gilles and LeRoy (1998), who address, many of the issues in a related setting.

[12] See Williams (1991).

[13] For example, $T < \infty$ and $t_i = T\,(1 - (1/2)^i)$.

[14] Recall, $K = X_+ \setminus \{0\}$, so that $x \in K$ if $x_i \ge 0$ for all $i \in \mathbb{N}$ and $x_i > 0$ for at least one $i \in \mathbb{N}$.

for $i \in \mathbb{N}$, where f_ω and g_i are finite and $S_0(\omega) = g_0 = 1$.[15] However, the sequences $\{f_i\}$ and $\{g_i\}$ are not necessarily bounded. Note that $S_i(\omega)$ is constant on each of the sets in R_i and thus measurable with respect to $\mathcal{F}_i$, so that $\{S_i\}$ is adapted to the filtration.[16]

For fixed i, $S_i(\omega)$ is a random variable; for fixed ω, $S_i(\omega)$ is a 'path'. In the following matrix, the rows are the random variables and columns are the stock price paths:

$$\begin{array}{c|cccccc} & & & \omega & & & \\ & 1 & 2 & 3 & 4 & 5 & \cdots \\ \hline t_0 & 1 & 1 & 1 & 1 & 1 & \cdots \\ t_1 & f_1 & g_1 & g_1 & g_1 & g_1 & \cdots \\ t_2 & f_1 & f_2 & g_2 & g_2 & g_2 & \cdots \\ t_3 & f_1 & f_2 & f_3 & g_3 & g_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array} \tag{3.3}$$

Every price path starts at one (the value of the stock before the first flip). We may think of (3.3) in terms of a tree structure:

$$\begin{array}{ccccc} t_0 & 1 & & & \\ & \downarrow\searrow & & & \\ t_1 & f_1 & g_1 & & \\ & \downarrow & \downarrow\searrow & & \\ t_2 & f_1 & f_2 & g_2 & \\ & \downarrow & \downarrow & \downarrow\searrow & \\ t_3 & f_1 & f_2 & f_3 & g_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots\ddots \end{array} \tag{3.4}$$

On each path, $S_i(\omega)$ converges to f_ω. More formally, define $S_\infty(\omega) := f_\omega$ and note that $S_i \xrightarrow{a.s.} S_\infty$, since

$$\lim_{i\to\infty} S_i(\omega) = S_\infty(\omega) \quad \text{for all } \omega \in \mathbb{N}.$$

Even though $S_i \in \ell^\infty(P)$ for all $i \in \mathbb{N}$, if $\{f_i\}$ is unbounded, $S_\infty \notin \ell^\infty(P)$.

Self-financing trading strategies. A trading strategy is a sequence of pairs $\{(\theta_i^B, \theta_i^S)\}$ adapted to the filtration, where $\theta_i^B(\omega)$ is the number of bonds held at time t_i in state ω and $\theta_i^S(\omega)$ is the number of shares of stock held at time t_i in state ω. The interpretation is that θ_i is constant on the half-open interval $[t_i, t_{i+1})$, while the $(i+1)$-st spin of the wheel occurs sometime in the open interval (t_i, t_{i+1}). A self-financing trading strategy is a trading strategy that satisfies the self-financing condition:

$$\theta_i^B - \theta_{i-1}^B + \left(\theta_i^S - \theta_{i-1}^S\right) S_i = 0, \quad \text{for } i = 1, 2, 3, \cdots. \tag{3.5}$$

[15] To allow for a non-zero interest rate r, let $\widetilde{B}_i(\omega) = e^{r\,t_i}$ be the money-market account and define $\widetilde{S}_i := \widetilde{B}_i\, S_i$. Then $B_i = \widetilde{B}_i/\widetilde{B}_i$ and $S_i = \widetilde{S}_i/\widetilde{B}_i$.

[16] Indeed, any stochastic process adapted to the filtration must have a similar structure as the stock price.

In other words, any change in the value of the stock holdings that comes from rebalancing must be offset by an equal change in the opposite direction of the value of the bond holdings. We refer to the value of the self-financing portfolio at time t_i as the *gain*: $G_i = \theta_i^B + \theta_i^S\, S_i$. Using the self-financing condition (3.5), we can express the dynamics of the gain as

$$G_i = G_0 + \sum_{j=1}^{i} \theta_{j-1}^S\, \Delta S_j, \tag{3.6}$$

where $G_0 = \theta_0^B + \theta_0^S\, S_0 = \theta_0^B + \theta_0^S$ is the initial cost of the self-financing trading strategy and the change in the stock price is

$$\Delta S_i(\omega) := S_i(\omega) - S_{i-1}(\omega) = \begin{cases} 0 & \omega < i \\ f_i - g_{i-1} & \omega = i \\ g_i - g_{i-1} & \omega > i. \end{cases}$$

Thus, conditional on the specified dynamics for the stock and bond prices, the gain for a self-financing trading strategy is completely specified by the initial investment G_0 and the sequence of stock holdings $\{\theta_j^S(\omega)\}_{j=0}^{\infty}$.

The value of the stock (and hence the value of the portfolio) does not change once 'red' first occurs, and thus the amount invested in the stock is irrelevant at that point. Therefore, for simplicity we can specify

$$\theta_i^S(\omega) = \begin{cases} 0 & \omega \le i \\ \xi_i & \omega > i. \end{cases}$$

With this normalization, a self-financing trading strategy is completely specified by $(G_0, \{\xi_j\}_{j=0}^{\infty})$. The gain generated by $(G_0, \{\xi_j\}_{j=0}^{\infty})$ is

$$G_i(\omega) = \begin{cases} G_\infty(\omega) & \omega \le i \\ c_i & \omega > i, \end{cases} \tag{3.7}$$

where

$$G_\infty(\omega) := G_0 + \xi_{\omega-1}\,(f_\omega - g_{\omega-1}) + \sum_{j=1}^{\omega-1} \xi_{j-1}\,(g_j - g_{j-1}) \tag{3.8a}$$

$$c_i := G_0 + \sum_{j=1}^{i} \xi_{j-1}\,(g_j - g_{j-1}). \tag{3.8b}$$

By construction, $G_i \xrightarrow{a.s.} G_\infty$.

We can design a trading strategy to generate $G_\infty = x \in X$ if the 'spanning condition' holds: $f_i \neq g_{i-1}$ for all $i \in \mathbb{N}$. The trading strategy that generates this gain is

$$\xi_i = \sum_{j=1}^{i+1} \alpha_j^i\,(G_\infty(j) - G_0), \tag{3.9}$$

where

$$\alpha_j^i = \begin{cases} \frac{(g_{j-1}-g_j)\prod_{s=j+1}^{i}(f_s-g_s)}{\prod_{s=j}^{i+1}(f_s-g_{s-1})} & j \le i \\ \frac{1}{f_{i+1}-g_i} & j = i+1. \end{cases}$$

The preceding trading strategy involves a fundamental indeterminacy: G_0 is a free parameter in (3.9). As a consequence of this indeterminacy, any G_∞ can be paired with any G_0. In other words, there is no connection between the initial investment at time 0 and the payout on the terminal date T that the gain process converges to almost surely. In particular, the doubling strategy is a gain process for which $G_0 = 0$ and $G_\infty(\omega) = 1$, which satisfies (3.9).

To remove the indeterminacy, we consider only those trading strategies that have a finite number of non-zero ξ_i. (We are no longer able to choose $G_\infty = x \in X$ arbitrarily.) Let i^* be the maximum index of the set of non-zero ξ_i and let $n = i^*+1$. Then $\xi_i = 0$ for all $i \ge n$. In addition, $G_i = G_\infty = G_n$ for all $i \ge n$. In effect, all proceeds are frozen at time t_n and carried forward to time T in the money-market account. We will refer to a gain with this property as a gain stopped at time t_n (or simply a stopped gain). For a stopped gain, $G_i \to G_\infty$ in all modes of convergence.

There is no indeterminacy for a gain stopped at t_n. Applying (3.9) to $\xi_n = 0$ shows that G_0 is uniquely determined by $\{G_\infty(j)\}_{j=1}^{n+1}$:

$$G_0 = \sum_{j=1}^{n+1} \left(\frac{\alpha_j^n}{\sum_{s=1}^{n+1} \alpha_s^n} \right) G_\infty(j) = \sum_{j=1}^{n} \mathit{\Gamma}_j \, G_\infty(j) + \left(\frac{\mathit{\Gamma}_{n+1}}{k_{n+1}} \right) G_\infty(n+1), \tag{3.10}$$

assuming $f_j \ne g_j$ (in addition to $f_j \ne g_{j-1}$), where

$$\mathit{\Gamma}_j := \frac{\alpha_j^n}{\sum_{s=1}^{n+1} \alpha_s^n} = k_j \prod_{s=1}^{j-1} (1 - k_s) \quad \text{and } k_j := \frac{g_j - g_{j-1}}{g_j - f_j}. \tag{3.11}$$

Since the gain process is stopped at time n, $G_\infty(n+1)$ is the payout for all $\omega \ge n+1$. Given the form of $\mathit{\Gamma}_j$ in terms of $\{k_s\}_{s=1}^{j}$, we have

$$\sum_{\omega=1}^{n} \mathit{\Gamma}_\omega + \frac{\mathit{\Gamma}_{n+1}}{k_{n+1}} = 1. \tag{3.12}$$

Consequently,

$$\lim_{n\to\infty} \frac{\mathit{\Gamma}_{n+1}}{k_{n+1}} = 1 - \sum_{\omega=1}^{\infty} \mathit{\Gamma}_\omega. \tag{3.13}$$

Let the gain stopped at t_n equal the payout of the n-th Arrow–Debreu security, $G_n = \delta_n$. From (3.10) we see that $\mathit{\Gamma}_n$ is the initial investment for this trading strategy, which we will refer to as the price of an Arrow–Debreu security. The k_i are 'conditional' state prices: Given $\omega > i$, the price of an Arrow–Debreu security in state $i+1$ is k_{i+1}. Arrow–Debreu security prices are all positive if and only if

$$0 < k_j < 1 \quad \text{for all } j \in \mathbb{N}. \tag{3.14}$$

The price of a unit tail starting at $n+1$ is given by $\mathit{\Gamma}_{n+1}/k_{n+1}$. Therefore, the positivity of the Arrow–Debreu security prices is sufficient for the positivity of all tails. Given (3.13) and (3.14), we have $\sum_{\omega=1}^{\infty} \Gamma_\omega \leq 1$ or equivalently $\Gamma_{i+1}/k_{i+1} \geq \sum_{\omega=i+1}^{\infty} \Gamma_\omega$. If, in addition, $\{k_i\}$ is bounded away from zero, then $\sum_{\omega=1}^{\infty} \Gamma_\omega = 1$, in which case $\Gamma_{i+1}/k_{i+1} = \sum_{\omega=i+1}^{\infty} \Gamma_\omega$.

State-price deflator. We define the state-price process as follows:[17] $Y_0 = 1$ and

$$Y_i = \prod_{j=1}^{i} Z_i, \quad \text{where } Z_i(\omega) := \begin{cases} 1 & \omega < i \\ \frac{k_i}{q_i} & \omega = i \\ \frac{1-k_i}{1-q_i} & \omega > i. \end{cases} \tag{3.15}$$

We require, as part of the definition of the state-price process, the strict positivity of $Y_i(\omega)$.[18] Thus, a state-price process exists if and only if the Arrow–Debreu security prices are all positive, the condition for which is given by (3.9). We can express the state-price process as follows:

$$Y_i(\omega) = \begin{cases} y(\omega) & \omega \leq i \\ h_i & \omega > i, \end{cases} \tag{3.16}$$

where

$$y(\omega) := \frac{\Gamma_\omega}{P_\omega} \quad \text{and} \quad h_i := y(i+1)\left(\frac{q_{i+1}}{k_{i+1}}\right).$$

Note that $E^P_{i-1}[Z_i] = 1$ for all i and all ω, and therefore $E^P_{i-1}[Y_i] = Y_{i-1}\, E^P_{i-1}[Z_i] = Y_i$. Moreover, given the positivity of Y_i, $E^P[|Y_i|] = E^P[Y_i] = 1$ for all $i \in \mathbb{N}$. Consequently, (i) Y_i is a martingale and (ii) the sequence $\{Y_i\}$ is bounded in $\ell^1(P)$. Martingale convergence guarantees $\{Y_i\}$ converges almost surely (and hence in measure) to an element in $\ell^1(P)$. Indeed, we see from (3.16) that $Y_i \xrightarrow{a.s.} y$. Whether $\{Y_i\}$ converges to y in τ_1^P depends on whether $\{Y_i\}$ is uniformly integrable (UI). In our setting, the following five condtions are equivalent:[19]

(1) $\{Y_i\}$ is UI
(2) $Y_i \xrightarrow{\tau_1^P} y$
(3) $Y_i = E^P_i[y]$ for all $i \in \mathbb{Z}_+$
(4) $E^P[y] = \lim_{i\to\infty} E^P[Y_i] = 1$
(5) $\sum_{\omega=1}^{\infty} \Gamma_\omega = 1$.

To summarize, if a uniformly integrable state-price process exists, it is given by (3.16) where the following holds:

$$\Gamma_\omega > 0 \quad \text{for all } \omega \in \mathbb{N} \qquad \text{and} \qquad \sum_{\omega=1}^{\infty} \Gamma_\omega = 1. \tag{3.17}$$

[17] See Appendix B for a method of computing the state-price process from the dynamics of the stock price.

[18] To allow for a non-zero interest rate r, let $\widetilde{B}_i(\omega) = e^{r\, t_i}$ be the money-market account and define $\widetilde{Y}_i := Y_i/\widetilde{B}_i$. Then $Y_i = \widetilde{Y}_i\, \widetilde{B}_i$.

[19] In Condition (3), $\{Y_i\}$ is said to be *closed* by y.

Deflated gains are martingales. Define the deflated stock and bond prices: $\hat{S}_i := S_i\,Y_i$ and $\hat{B}_i := B_i\,Y_i = Y_i$. The state-price process has the property that the deflated asset prices are martingales: $E^P_{i-1}[\hat{S}_i] = \hat{S}_{i-1}$ and $E^P_{i-1}[\hat{B}_i] = \hat{B}_{i-1}$ (which amounts to $E^P_{i-1}[Y_i] = Y_{i-1}$). The deflated gain, $\hat{G}_i := G_i\,Y_i = \theta^B_i\,\hat{B}_i + \theta^S_i\,\hat{S}_i$, inherits the martingale property: $E^P_{i-1}[\hat{G}_i] = \hat{G}_{i-1}$.

It remains to show $E^P_{i-1}[\Delta\hat{S}_i] = 0$, where $\Delta\hat{S}_i := \hat{S}_i - \hat{S}_{i-1}$. Note that

$$\Delta\hat{S}_i = Y_{i-1}\,\big(Z_i\,\Delta S_i + (1 - Z_i)\,S_{i-1}\big),$$

and therefore $E^P_{i-1}[\Delta\hat{S}_i] = Y_{i-1}\,E^P_{i-1}[Z_i\,\Delta S_i]$. Since $\Delta S_i = 0$ for $\omega \le i-1$, it is enough to show that $E^P_{i-1}[Z_i\,\Delta S_i] = 0$ for $\omega > i-1$, in which case

$$E^P_{i-1}[Z_i\,\Delta S_i] = q_i\left(\frac{k_i}{q_i}\right)(f_i - g_{i-1}) + (1-q_i)\left(\frac{1-k_i}{1-q_i}\right)(g_i - g_{i-1}) = 0, \tag{3.18}$$

where we have used the conditional probabilities q_i and $1-q_i$ in computing the conditional expectation.

Equivalent measure. Proposition: There exists an equivalent measure if and only if there exists a uniformly-integrable state-price deflator. Proof of the 'if' part: If $\{Y_i\}$ is UI, then $Y_i \xrightarrow{\tau^P_1} y \in \ell^1(P)$ and $E^P[y] = 1$. These conditions guarantee the existence of an equivalent measure Q, with Radon–Nikodym derivative $dQ/dP = y$, so that $Q_\omega := Q[\{\omega\}] = y(\omega)\,P[\{\omega\}] = \mathit{\Gamma}_\omega$ for all $\omega \in \mathbb{N}$. Conversely, if there exists an equivalent measure, then there is a Radon–Nikodym derivative $y \in \ell^1(P)$, such that y is strictly positive and $E^P[y] = 1$. Then $Y_i = E^P_i[y]$ is a UI state-price process.

Arbitrages and free lunches. In order to apply Kreps' definitions of arbitrage and free lunches to the dynamic setting, we need a correspondence between the space of stopped gains and the marketed subspace.

The marketed subspace. Here we define the marketed subspace and a linear pricing functional on it. We show that the space corresponds the set of stopped gain processes of self-financing trading strategies and the pricing functional corresponds to the initial investment.

Let the set of marketed securities be given by $M_0 = \{B\} \cup \{S_i : i \in \mathbb{N}\}$. Then $m \in M$ (the marketed subspace) if and only if $m = \eta_0\,B + \sum_{i=1}^{i^*} \eta_i\,S_i$ for finite i^*. Note that

$$m(\omega) = \eta_0 + \sum_{j=1}^{\omega-1} \eta_j\,g_j + f_\omega \sum_{j=\omega}^{i^*} \eta_j, \tag{3.19}$$

where $\eta_j = 0$ for $j > i^*$. Let $\pi_0(B) = \pi_0(S_i) = 1$. For $m \in M$, define $\pi(m) := \eta_0\,\pi_0(B) + \sum_{i=1}^{i^*} \eta_i\,\pi_0(S_i) = \sum_{i=0}^{i^*} \eta_i$.

For any $m \in M$, define the following self-financing trading strategy: $G_0 = \pi(m)$ and $\xi_i = \sum_{j=i+1}^{i^*} \eta_j$. This trading strategy produces $G_\infty(\omega) = m(\omega)$. In addition, since $\xi_i = 0$ for $i \ge n = i^* + 1$, the gain is stopped at t_n, so that $G_j = G_n = m$ for all $j \ge n$. Conversely, for any self-financing trading strategy that is stopped at t_n, there is an $m \in M$, where $m = G_n$ is given by (3.19) such that $\eta_0 = G_0 - \xi_0$ and $\eta_i = \xi_{i+1} - \xi_i$. Moreover, $\pi(m) = G_0$.

The preceeding correspondence establishes the following two facts: $m \in M \cap K \iff G_n \in K$ and $\pi(m) > 0 \iff G_0 > 0$.

Arbitrage opportunities. Fix a space $X \supseteq M$ and a cone $K = X_+ \setminus \{0\}$. The Krepsian statement of no arbitrage opportunities is $m \in M \cap K \implies \pi(m) > 0$. This is equivalent to $G_n \in K \implies G_0 > 0$ where G_n is any stopped gain process generated by a self-financing trading strategy.

There are no arbitrage opportunities given a state-price process: Given $Y_n > 0$ and $E^P[G_n Y_n] = G_0$, we conclude $G_n \in K \implies G_0 > 0$. Conversely, if there are no arbitrage opportunities, we can construct a state-price process.

Free lunches. Fix a topological space (X, τ) where $X \supseteq M$ and the positive cone $K = X_+ \setminus \{0\}$. Applying the Krepsian analysis, there are no free lunches if π extends to $\psi \in \Psi$. We show that if there exists a state-price process that has limit points in X^*, then there exists a $\psi \in \Psi$ that extends π to of all X.

Consider $(X, \tau) = (\ell^p(P), \tau_p^P)$, where $1 \leq p < \infty$, for which the dual space is $(X^*, \tau^*) = (\ell^q(P), \tau_q^P)$, where $q = p/(p-1)$. If $Y_i \xrightarrow{\tau^*} y \in X^*$, then $\psi(x) = E^P[x\, y]$ for all $x \in X$ and $\psi(m) = \pi(m)$ for all $m \in M$.

Now consider $(X, \tau) = (\ell^1(P)^{**}, w^*)$, for which $(X^*, \tau^*) = (\ell^\infty(P), \tau_\infty^P)$. If $Y_i \xrightarrow{\tau^*} y \in X^*$, then for all $x \in X$, where $x = x^1 + x^\perp$ is the Yosida–Hewitt decomposition, $\psi(x) = E^P[x^1\, y] + \int_\Omega y\, d\varphi$, where $\varphi \in \boldsymbol{pa}$ is the pure charge that corresponds to $x^\perp \in \ell^1(P)^\perp$, and $\psi(m) = E^P[m\, y] = \pi(m)$ for all $m \in M \subseteq \ell^1(P)$.

Finally consider $(X, \tau) = (\ell^\infty(P), \tau_\infty^P)$, for which $(X^*, \tau^*) = (\ell^1(P)^{**}, w^*)$. Since $\{Y_i\}$ is bounded in $\ell^1(P)$ and $Y_i \xrightarrow{P} y \in \ell^1(P)$, there exist limit points of $\{Y_i\}$ in $\ell^1(P)^{**}$, all of which have the form $y + \Upsilon^\perp$, where $\Upsilon^\perp \in \ell^1(P)^\perp$. (If $Y_i \xrightarrow{\tau_1^P} y$, then y is the only limit point and $\Upsilon^\perp \equiv 0$.) Then for all $x \in X$, $\psi(x) = E^P[x\, y] + \int_\Omega x\, d\Phi$, where $\Phi \in \boldsymbol{pa}$ is the pure charge that corresponds to $\Upsilon^\perp$, and $\psi(m) = \pi(m)$ for all $m \in M$.

Let $\{G_i\}_{i=1}^\infty$ be any gain generated by a self-financing trading strategy and let $\{G_\alpha\}_{\alpha \in A}$ be any subnet of $\{G_i\}_{i=1}^\infty$. If there are no free lunches, then $G_\alpha \xrightarrow{\tau} x \in K \implies G_0 > 0$.

The doubling strategy. The doubling strategy is characterized by $G_0 = 0$ and $G_\infty(\omega) = 1$. Referring to (3.9), the trading strategy that generates this gain is

$$\xi_i = \sum_{j=1}^{i+1} \alpha_j^i = \left(\frac{1}{f_1 - 1}\right) \prod_{j=1}^{i} \frac{f_j - g_j}{f_{j+1} - g_j} = \left(\sum_{j=1}^{i} f_j\, \Gamma_j + \frac{f_{i+1}\, \Gamma_{i+1}}{k_{i+1}} - 1\right)^{-1}. \quad (3.20)$$

The trading strategy involves taking an initial position in the stock of $1/(f_1 - 1)$ and financing it with the bond so that no funds are invested. The first time 'red' appears, the stock position is closed and all proceeds are invested in the bond thereafter. If the first 'red' has not appeared after the i-th spin, the position in the stock is increased by the factor $(f_i - g_i)/(f_{i+1} - g_i)$, 'doubling' as it were. The boundedness of $\{\xi_i\}$ depends on $S_\infty(\omega) = f_\omega$ [see the rightmost expression for ξ_i in (3.20)]. For example, if $f_1 = a > 1$ and $f_i = 1/a$ for $i \geq 2$, then

$$\xi_0 = \frac{1}{a-1} \quad \text{and} \quad \xi_i = \frac{a}{(a-1)\big((1+a)\,k_1 - 1\big)} \quad \text{for } i \geq 1. \tag{3.21}$$

In this case, the amount invested in the stock does not change after the second spin of the wheel. It is important to recognize that the boundedness of the trading strategy $\{\xi_i\}$ plays no role in determining the existence of arbitrage opportunities or free lunch opportunities.

Note that $G_i = z_i$ for all $i \in \mathbb{N}$, where z_i is given in (2.8), where $c_i = 1 - \beta_i$. Therefore, we have already examined the properties of $\|G_i\|_p^P$. The only novelty here is to recognize $\beta_i = k_{i+1}/\Gamma_{i+1}$, so that for $1 \leq p < \infty$,

$$\|1 - G_i\|_p^P = \frac{k_{i+1}}{\Gamma_{i+1}} \left(\frac{P_{i+1}}{q_{i+1}}\right)^{1/p},$$

$$\|G_i\|_1^P = \sum_{\omega=1}^{i} P_\omega + h_i^{-1} \sum_{\omega=1}^{i} \Gamma_\omega,$$

and

$$\lim_{i\to\infty} \|G_i\|_1^P = \lim_{i\to\infty} 1 + h_i^{-1}.$$

where h_i is given following (3.16).

The suicide strategy. The suicide strategy is characterized by $G_0 = 1$ and $G_\infty(\omega) = 0$. The trading strategy that generates this gain is simply the negative of that given in (3.20).

The buy-and-hold strategy. The buy-and-hold self-financing trading strategy for the stock is characterized by $G_0 = 1$ and $\gamma_j = 1$ for all $j \in \mathbb{Z}_+$. The gain generated by this trading strategy is $G_i = 1 + \sum_{j=1}^{n} \Delta S_j = S_i$. As noted above, $S_i \xrightarrow{a.s.} S_\infty$, where $S_\infty(\omega) = f_\omega$. Two questions arise. First, is $S_\infty \in X$? By construction, S_i is bounded for every $i \in \mathbb{N}$. Nevertheless, S_∞ need not be bounded, in which case $S_\infty \notin \ell^\infty(P)$. For $p < \infty$, the ℓ^p norm of S_∞ is given by

$$\|S_\infty\|_p^P = \left(\sum_{\omega=1}^{\infty} |f_\omega|^p \, P_\omega\right)^{1/p},$$

which may or may not be finite. Second, assuming $S_\infty \in X$, does $S_i \xrightarrow{\tau} S_\infty$? For example, given $(X, \tau) = (\ell^p(P), \tau_p^P)$, consider

$$\lim_{i\to\infty} \|S_i - S_\infty\|_p^P = \lim_{i\to\infty} \left(\sum_{\omega=i+1}^{\infty} |g_i - f_\omega|^p \, P_\omega\right)^{1/p},$$

which may or may not be zero.

Examples. We revisit the two examples in the dynamic setting. Recall $P_\omega = \rho\,(1-\rho)^{\omega-1}$, where $0 < \rho < 1$.

Example 1. Recall $\mathit{\Gamma}_i = \zeta\,(1-\zeta)^{i-1}$. In this case, $k_i = \zeta$. Since $\sum_{i=1}^{\infty} \mathit{\Gamma}_i = 1$, $\{Y_i\}$ is UI and $Y_i \xrightarrow{\tau_1^P} y$, where $y(\omega) = (\zeta/\rho)\left((1-\zeta)/(1-\rho)\right)^{\omega-1}$.[20]

Let the stock-price dynamics be given by

$$f_i = \left(1 + \mu\left(\frac{1-\zeta}{\zeta}\right)\right)(1-\mu)^{i-1}, \quad \text{and} \quad g_i = (1-\mu)^i.$$

where $0 < \zeta < 1$ and $-\zeta/(1-\zeta) \le \mu < 1$, $\mu \ne 0$. In words (assuming $\mu > 0$), the stock price falls by $100\,\mu$ percent until 'red' first occurs, at which point it rises by $100\,(\mu/\zeta)\,(1-\zeta)$ percent and does not change thereafter. In this case, the doubling-strategy trading strategy is

$$\xi_i = \frac{1}{\mu}\left(\frac{\zeta}{1-\zeta}\right)\left(\frac{1}{(1-\zeta)\,(1-\mu)}\right)^i,$$

which is unbounded.

If $\mu < 0$, $S_\infty \notin \ell^\infty$ (even though every element in the sequence is in ℓ^∞) since S_∞ is not bounded in that case. The $\ell^p(P)$ norm of S_∞ is given by

$$\|S_\infty\|_p^P = \left(\frac{\left(1+\mu\left(\frac{1-\zeta}{\zeta}\right)\right)^p \rho}{1-(1-\mu)^p\,(1-\rho)}\right)^{1/p} < \infty, \quad \text{if } \rho > 1-(1-\mu)^{-p}.$$

This condition does not impose a constraint unless $\mu < 0$. In order for both the no-free-lunch condition and the condition for $S_\infty \in \ell^p(P)$ to hold, we must have $1-(1-\mu)^{-p} < \rho < 1-(1-\zeta)^p$. For $\mu = -(1-\zeta)/\zeta$, one of the two conditions must be violated; in addition, $S_\infty = 0$, so that the buy-and-hold strategy is the suicide strategy.

Now let the stock-price dynamics be given as follows: $f_1 = a > 1$, $f_i = 1/a$ for $i \ge 2$, and

$$g_i = \frac{1+(a-1)\,(1-\zeta)^i}{a}.$$

The trading strategy that generates the doubling strategy, which is given by (3.21), is bounded. Note that

$$\|S_i - S_\infty\|_p^P = \left(\frac{a-1}{a}\right)\left(\frac{(1-\rho)^{1/p}}{1-\zeta}\right)^i \quad \text{for } i \ge 2.$$

Hence, the condition for $S_i \xrightarrow{\tau} S_\infty$ is $\rho = \zeta$ or $\rho < 1-(1-\zeta)^p$. In other words, S_∞ is in the space if and only if there are no free lunches.

Example 2. Recall, $\mathit{\Gamma}_i = \left((2\,i)\,(i+1)\right)^{-1} > 0$ and $\sum_{\omega=1}^{\infty} \mathit{\Gamma}_\omega = 1/2$. The state-price process is not UI. Note that $k_i = (1+i)^{-2}$. The doubling-strategy gain is characterized by $c_i = -i/(i+2)$, which is bounded. Of course, $G_i \xrightarrow{\tau_p^P} 1$ for $1 \le p < \infty$, but $\{G_i\}$ does not converge in τ_∞^P, for which there are no free lunches.

[20] For $\omega > i$, the price of risk is $\lambda_i(\omega) = (\rho-\zeta)\left(\rho\,(1-\rho)\right)^{-1/2}$. See Appendix B.

Back and Pliska (1991) present an example that fits into this case.[21, 22] They choose the following stock-price dynamics: $f_i = 2^{-i}\,(i^2 + 2\,i + 2)$ and $g_i = 2^{-i}$. For this choice of stock-price dynamics, the doubling-strategy trading strategy is

$$\xi_i = \frac{2^{2+i}}{i^2 + 5\,i + 6},$$

which is unbounded. Of course, we can choose different stock-price dynamics: $f_1 = a > 0$, $f_i = 1/a$ for $i \geq 2$, and

$$g_i = \frac{1 - i + (4 - a)\ a\ (1 + i)}{2\,a\ (2 + i)}.$$

For these stock-price dynamics, the doubling-strategy trading strategy, which is given by (3.21), is bounded.

Back and Pliska define arbitrage opportunities differently than defined here. According to their definition, there are no arbitrage opportunities in their example. A necessary component of the absence of arbitrage opportunities is $\lim_{i\to\infty}\ f_i = 0$. Thus, our modification of their example is subject to arbitrage opportunities according to their definition because $\lim_{i\to\infty}\ f_i = 1/a > 0$. In terms of our definitions, the free lunches available in $\ell^p(P)$ for $p < \infty$ do not require unbounded trading strategies $\{\xi_i\}$. Thus, according to their definitions, our example presents the following situation in $(\ell^\infty(P), \tau^P_\infty)$: There are arbitrage opportunities, but their are no free lunches.

Appendix A. Mathematical preliminaries

Take as given a measure space $(\Omega, \mathcal{F}, \mu)$, where $\Omega = \mathbb{N} := \{1, 2, 3, \cdots\}$ is the set of natural numbers, $\mathcal{F} = \mathfrak{P}(\mathbb{N})$ is the set of all subsets of $\mathbb{N}$, and μ is a finite measure characterized by $\mu[\{\omega\}] > 0$ for all $\omega \in \mathbb{N}$, where $\mu[\mathbb{N}] < \infty$. For the space of commodity bundles (security payouts), let $X = \ell^p(\mu) = L^p(\mathbb{N}, \mathfrak{P}(\mathbb{N}), \mu)$, for $1 \leq p \leq \infty$.[23, 24] The $\ell^p(\mu)$ spaces of sequences are characterized as follows: $x \in \ell^p(\mu)$ if $\|x\|^\mu_p < \infty$, where

$$\|x\|^\mu_p = \begin{cases} \Big(\sum_{\omega=1}^\infty |x(\omega)|^p\,\mu[\{\omega\}]\Big)^{1/p} & 1 \leq p < \infty \\ \sup_\omega\ |x(\omega)| & p = \infty. \end{cases}$$

[21]Back and Pliska do not address viability *per se*. Instead, they consider the existence of an 'optimal demand.' For an agent constrained to consume in the positive orthant, an optimal demand can exist even absent viability when the agent is driven to a corner. Thus, Back and Pliska find an optimal demand exists in ℓ^p for $p < \infty$, even though the pair (M, π) is not viable.

[22]Their W corresponds to our G_0, their ϕ_i corresponds to our ξ_{i-1}, and they set $\rho = 1/6$.

[23]The only set in the σ-algebra $\mathfrak{P}(\mathbb{N})$ with measure zero is the null set $\varnothing$. As a consequence, given $x, y \in X$, $x = y$ *almost everywhere* if and only if $x(\omega) = y(\omega)$ for *every* $\omega \in \mathbb{N}$.

[24]Our use of ℓ^p is a bit nonstandard. Ordinarily, the measure is restricted to the *counting* measure. In fact, we exclude it because it is not finite.

Let τ_p^μ denote the $\ell^p(\mu)$ norm topology. Convergence in the $\ell^p(\mu)$ norm of a sequence $\{x_i\}$ to an element x (denoted $x_i \xrightarrow{\tau_p^\mu} x$) is characterized by

$$\lim_{i\to\infty} \|x - x_i\|_p^\mu = 0.$$

If $\mu[\mathbb{N}] = 1$, then μ is a probability measure, and the expectation operator is defined by $E^\mu[x] := \sum_{\omega=1}^\infty x(\omega)\,\mu[\{\omega\}]$ for $x \in \ell^1(\mu)$.

In all cases, let $K = X_+ \setminus \{0\}$ be the positive cone of X with the origin deleted; i.e., $x \in K$ if $x_\omega \geq 0$ for all $\omega \in \mathbb{N}$ and $x_\omega > 0$ for at least one ω.[25] Consequently, it is sufficient to specify a topological space (X, τ).

Let $\ell^p(\mu)^* = (\ell^p(\mu), \tau_p^\mu)^*$ denote the topological dual space of $\ell^p(\mu)$. The elements of $\ell^p(\mu)^*$ are continuous linear functionals on $\ell^p(\mu)$. For $1 \leq p < \infty$, $\ell^p(\mu)^* = \ell^q(\mu)$, where $1/p + 1/q = 1$; for $1 < p < \infty$, the bi-dual (i.e., the dual of the dual) of $\ell^p(\mu)$ is the space itself: $\ell^p(\mu)^{**} = \ell^p(\mu)$.

Note that $\ell^1(\mu)^* = \ell^\infty(\mu)$ and therefore $\ell^1(\mu)^{**} = \ell^\infty(\mu)^*$ However, $\ell^\infty(\mu)^* \neq \ell^1(\mu)$ and hence $\ell^1(\mu)^{**} \neq \ell^1(\mu)$. Nevertheless, $\ell^1(\mu) \subset \ell^\infty(\mu)^* = \ell^1(\mu)^{**}$. In fact, every $x \in \ell^1(\mu)^{**}$ has the unique decomposition $x = y + z$, where $y \in \ell^1(\mu)$ and $z \in \ell^1(\mu)^\perp$, where $\ell^1(\mu)^\perp$ denotes the orthogonal complement of $\ell^1(\mu)$ in $\ell^1(\mu)^{**}$.[26] We can make the following three identifications: (1) identify $\ell^1(\mu)^{**}$ with $ba(\mathfrak{P}(\mathbb{N}))$, the space of all signed charges (*i.e.,* finitely-additive set functions) of bounded variation on the σ-algebra $\mathfrak{P}(\mathbb{N})$ of all subsets of $\mathbb{N}$; (2) identify $\ell^1(\mu)$ with $ca(\mathfrak{P}(\mathbb{N}))$, the band of all σ-additive signed measures in $ba(\mathfrak{P}(\mathbb{N}))$; and (3) identify $\ell^1(\mu)^\perp$ with $pa(\mathfrak{P}(\mathbb{N}))$, the band of all purely finitely additive signed measures of $ba(\mathfrak{P}(\mathbb{N}))$.[27] Let $\tau = w^*$ be the weak* topology. With this topology, the topological dual of $\ell^1(\mu)^{**}$ is its pre-dual $\ell^1(\mu)^* = \ell^\infty(\mu)$.

[Discuss weak and weak* convergence.]

[Discuss weak topology w_p^P.]

Define the indicator function:

$$\mathbf{1}_A(\omega) := \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A, \end{cases}$$

where A is some set. An Arrow–Debreu security pays one unit in one state of the world. The payout on the i-th Arrow–Debreu security (the security that pays 1 in state i) is $\delta_i(\omega) := \mathbf{1}_{\{i\}}(\omega)$.

[25] $X_+ := \{x : x_\omega \geq 0,\ \forall\,\omega \in \mathbb{N}\}$ and $X_{++} := \{x : x_\omega > 0,\ \forall\,\omega \in \mathbb{N}\}$.

[26] Let $X = \ell^1(\mu)^{**}$. Note that $x \in K = X_+ \setminus \{0\}$ implies $y \in X_+$ and $z \in X_+$ and at least one of the two not 0.

[27] See Aliprantis and Border (1999, Chapter 15) for further details.

Let τ_m denote the topology of convergence in measure.[28] The topology of convergence in measure is generated by the metric

$$d(f,g) = \int_\Omega \frac{|f-g|}{1+|f-g|}\,d\mu = \sum_{\omega=1}^{\infty} \left(\frac{|f(\omega)-g(\omega)|}{1+|f(\omega)-g(\omega)|} \right) \mu[\{\omega\}]. \tag{A.1}$$

In particular, $f_n \xrightarrow{\mu} f \iff d(f_n, f) \to 0$. Also note that $f_n \xrightarrow{a.s.} f \implies d(f_n, f) \to 0$, where $f_n \xrightarrow{a.s.} f$ is shorthand for

$$\lim_{n\to\infty} f_n(\omega) = f(\omega) \quad \text{for all } \omega \in \mathbb{N}.$$

Consider the topological space $(\ell^p(\mu), \tau_m)$, where $1 \le p \le \infty$. The problem with this space (with respect to viability) is that $\Psi = \varnothing$; in other words, the set of strictly positive continuous linear functionals on $(\ell^p(\mu), \tau_m)$ is empty. Here is a proof adapted from Kreps.[29] If ψ is a K strictly positive continuous linear functional, then $\psi(\delta_i) > 0$. Let $y_i = \delta_1 - [\psi(\delta_1)/\psi(\delta_i)]\,\delta_i$. By linearity $\psi(y_i) = 0$. Now consider

$$d(y_i, \delta_1) = \sum_{\omega=1}^{\infty} \left(\frac{\frac{\psi(\delta_1)}{\psi(\delta_i)}\delta_i(\omega)}{1+\frac{\psi(\delta_1)}{\psi(\delta_i)}\delta_i(\omega)} \right) \mu[\{\omega\}] = \left(\frac{\psi(\delta_1)}{\psi(\delta_1)+\psi(\delta_i)} \right) \mu[\{i\}] < \mu[\{i\}].$$

By the finiteness of μ, $\lim_{i\to\infty} d(y_i, \delta_1) = 0$ and therefore $y_i \xrightarrow{\mu} \delta_1$. Thus, if ψ is continuous it follows that $\psi(\delta_1) = 0$, a contradiction.

The linear vector space $\ell^0(\mu) = L^0(\mathbb{N}, \mathfrak{P}(\mathbb{N}), \mu)$ is the space of all μ-measurable sequences with the metric topology of convergence in measure.[30] L^0 is a Fréchet space (it is topologically complete), but it is not a Banach space (it does not have a norm).

Appendix B. Computing the state-price deflator from the stock-price dynamics

We can compute the price of risk and the state-price process from the dynamics of the stock price by using conditional expectations to decompose changes in the stock price into expected and unexpected components. The conditional expectation of the stock price next period is

$$E_i^P[S_{i+1}](\omega) = \begin{cases} f_\omega & \omega \le i \\ q_{i+1}\, f_{i+1} + (1-q_{i+1})\, g_{i+1} & \omega > i. \end{cases} \tag{B.1}$$

[28] Convergence in measure and L^0 are discussed in Aliprantis and Border (1999, Chapter 12, pp. 446–451). On a finite measure space (as we assume), pointwise convergence implies convergence in measure. Also, norm convergence in any $L^p(\mu)$ space implies convergence in measure μ. Therefore, if $\{f_n\}$ converges pointwise to f, then it converges in measure, and hence f is the only candidate for norm convergence.

[29] Kreps presents his proof (p. 24) in terms of pointwise convergence in the context of an example that illustrates the proposition that the absence of free lunches does not imply viability. In his example, there are no free lunches because the marketed subspace is one-dimensional.

[30] Neither ℓ^0 or ℓ^∞ depends on μ.

Define the expected change in the stock price, $m_i := E_i^P[\Delta S_{i+1}]$, where

$$m_i(\omega) = \begin{cases} 0 & \omega \le i \\ (q_{i+1} - k_{i+1})\,(f_{i+1} - g_{i+1}) & \omega > i, \end{cases}$$

and define the volatility of the change in the stock price

$$\sigma_i(\omega) := \begin{cases} 0 & \omega \le i \\ \sqrt{q_{i+1}\,(1 - q_{i+1})}\,(f_{i+1} - g_{i+1}) & \omega > i. \end{cases}$$

Now define the 'shock' process:

$$\Delta\varepsilon_i(\omega) := \begin{cases} 0 & \omega < i \\ \left(\frac{1-q_i}{q_i}\right)^{1/2} & \omega = i \\ -\left(\frac{q_i}{1-q_i}\right)^{1/2} & \omega > i, \end{cases}$$

where $E^P[\Delta\varepsilon_i] = 0$. In addition, $\Delta\varepsilon_i$ has the following conditional properties: $E_i^P[\Delta\varepsilon_{i+1}](\omega) = 0$ for all $\omega \in \mathbb{N}$ and

$$E_i^P[\Delta\varepsilon_{i+1}^2](\omega) = \begin{cases} 0 & \omega \le i \\ 1 & \omega > i. \end{cases}$$

Note the $\Delta\varepsilon_i$ are not independent, since $\Delta\varepsilon_i = 0$ implies $\Delta\varepsilon_{i+j} = 0$ for all $j \in \mathbb{N}$. Nevertheless, they are conditionally independent in the sense that, for $\omega > i$, future values are independent of current and past values. We express the change in the stock price as sum of the expected change plus the volatility of the change times the change in the shock:

$$\Delta S_i(\omega) = m_{i-1}(\omega) + \sigma_{i-1}(\omega)\,\Delta\varepsilon_i(\omega),$$

where

$$\sigma_{i-1}(\omega)\,\Delta\varepsilon_i(\omega) = \begin{cases} 0 & \omega < i \\ (1 - q_i)\,(f_i - g_i) & \omega = i \\ -q_i\,(f_i - g_i) & \omega > i. \end{cases}$$

Now construct the price of risk from the drift and diffusion:

$$\lambda_i(\omega) := \begin{cases} 0 & \omega \le i \\ \frac{m_i(\omega)}{\sigma_i(\omega)} = \frac{q_{i+1} - k_{i+1}}{\sqrt{q_{i+1}\,(1-q_{i+1})}} & \omega > i. \end{cases} \tag{B.2}$$

'Risk-neutrality' can be characterized by $q_i = k_i$ for all i. At this point, we can construct the state-price process using $Z_i = 1 - \lambda_{i-1}\,\Delta\varepsilon_i$.

In our definitions of the 'drift' and 'diffusion' of the stock price, we have not normalized per unit of time. We could easily do so, but the results would not change, since the differential 'dynamics' are derived from the given level processes. In particular, let

$$\Delta S_i = \widetilde{m}_{i-1}\,\Delta t_i + \widetilde{\sigma}_{i-1}\,\Delta\widetilde{\varepsilon}_i \quad \text{and} \quad Z_i = 1 - \widetilde{\lambda}_{i-1}\,\Delta\widetilde{\varepsilon}_i,$$

where $\Delta t_i := t_i - t_{i-1}$, $\widetilde{m}_{i-1} := m_{i-1}/\Delta t_i$, $\widetilde{\sigma}_{i-1} := \sigma_{i-1}/\sqrt{\Delta t_i}$, $\widetilde{\lambda}_{i-1} := \lambda_{i-1}/\sqrt{\Delta t_i}$, and $\Delta\widetilde{\varepsilon}_i := \sqrt{\Delta t_i}\,\Delta\varepsilon_i$. Although $\widetilde{m}_i$, $\widetilde{\sigma}_i$, and $\widetilde{\lambda}_i$ may be explosive, the properties of S_i, Z_i, and Y_i are unchanged.

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