

# WEAK-\* CONVERGENCE AND THE DOUBLING STRATEGY

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*Still a bit incomplete.*

ABSTRACT. The doubling strategy does not produce an arbitrage in the space of signed measures equipped with the weak-\* topology. (A signed measure represents a payout.) The absence of arbitrage opportunities is guaranteed by the existence of a valuation operator (a strictly positive continuous linear functional). Nevertheless, the sequence of signed measures generated by the doubling strategy produces what appears to be an arbitrage from the perspective of convergence almost everywhere of the corresponding sequence of Radon–Nikodym derivatives taken with respect to a fixed positive numeraire measure.

## INTRODUCTION [INCOMPLETE]

In their seminal paper, Harrison and Kreps (1979, p. 400) refer to

the well known doubling strategy by which one is sure to win at roulette: Bet \$1 on red, and keep doubling your bets until red comes out. To effect this strategy, you must be able to bet a countable number of times, although you will only bet a finite number of times in any particular state.

The doubling strategy is an arbitrage because each bet is costless, while the gain after  $n$  bets converges to \$1 with probability one as  $n$  goes to infinity. Side conditions typically solve this pathology by removing the doubling strategy from the choice set. Note, however, that in describing the situation as a pathology we have adopted the topology of convergence in probability (the topology of almost sure convergence has the same effect). With other topologies the doubling strategy does not necessarily converge to \$1, and restricting the strategy choice becomes unnecessary. There is no compelling reason to adopt the topology of convergence in probability; often, in fact, there are compelling reasons not to.

In this paper we present a framework in which the doubling strategy converges but not to an arbitrage. Although the framework is fairly general, we have not yet extended our results to continuous-time stochastic processes in any important way.

Our main point is that the choice of a topology dictates which sequences of payouts are arbitrages, a point made by Kreps (1981), who apparently did not

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apply it to the case of dynamic security market models.<sup>1</sup> We illustrate it with the traditional example of the doubling strategy cast in the simplest possible setting: A countable state space where the state of the world is simply the number of spins that it takes for the first RED to occur.

Our treatment of the doubling strategy is inspired by the suggestion in Gilles and LeRoy (1997, Section 6.3) that the doubling strategy could be modeled as a payout bubble.<sup>2</sup> Their paper in turn is an outgrowth of Bewley (1972). Our setup also encompasses the example in Back and Pliska (1991).<sup>3</sup> In passing, we mention the approach taken by Delbaen and Schachermayer (1994).

It is fruitful to think of payouts as continuous linear functionals on a space of price systems. (As such, the space of payouts is the topological dual space of the space of price systems.) A payout can be characterized by its ‘action’ over the entire space of price systems. The action of a payout on a price system is its value according to that price system. Two payouts that agree in value according to every price system can be identified. Similarly, convergence of a sequence of payouts can be evaluated in terms of sequences of values: A sequence of payouts converges to a given payout if and only if the sequence of values converges to the value of the given payout for every price system in the space. This notion of convergence is called weak-\* convergence.

Aliprantis and Border (1999, p. 163) write<sup>4</sup>

[W]e are led to the study of *dual pairs*  $\langle X, X^* \rangle$  of spaces and their associated *weak topologies*. . . . The weak topology on  $X^*$  induced by  $X$  is called the weak-\* topology. The most familiar example of a dual pair is probably the pairing of functions and measures—each defines a linear functional via the integral  $\int_X f d\mu$ , which is linear in  $f$  for fixed  $\mu$ , and linear in  $\mu$  for fixed  $f$ . (The weak topology induced on probability measures by this duality with continuous functions is the topology of convergence in distribution that is used in the Central Limit Theorems.) . . .

Debreu (1954) introduced dual pairs in economics in order to describe the duality between commodities and prices. According to

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<sup>1</sup>In fact, Kreps uses as an example the model we adopt here: the space of signed measures equipped with the weak-\* topology. However, the subject matter content (differentiated products markets) of the paper he cites [Mas-Colell (1975)] that use the model is somewhat different from what we propose. See also Jones (1984).

<sup>2</sup>Our analysis of the examples taken from Gilles and LeRoy (1997) is much simpler than what is given there because we have reduced the set of linear functionals relative to those treated in the model in that paper by assuming compactness of  $X$ . See below.

<sup>3</sup>Werner (1997) provides extensive treatment of the same static version of their example that we present here. In a related static model Gilles and LeRoy (1998) discuss many of the issues raised by Back and Pliska. Pliska (1997, Chapter 7) provides an introduction to these issues.

<sup>4</sup>We have made some minor adjustments to their notation. In particular, they use  $X'$  to denote the topological dual and  $X^*$  to denote the algebraic dual, whereas we adopt the more usual notation where  $X^*$  denotes the topological dual.

this interpretation, a dual pair  $\langle X, X^* \rangle$  represents the commodity-price duality, where  $X$  is the commodity space,  $X^*$  is the price space, and  $\langle x, x^* \rangle$  is the value of the bundle  $x$  at prices  $x^*$ . . . .

In this paper, we adopt the duality of functions and measures and apply it to the duality of prices and quantities; however we find it convenient to reverse the roles of the dual pair  $\langle X, X^* \rangle$ , letting the space of functions  $X$  represent the price space and the space of measures  $X^*$  represent the commodity space. We equip  $X^*$  with the weak-\* topology. This has the effect (among others) of making  $X$  the topological dual of  $X^*$ .

**Outline of the paper.** In Section 1 we present the results from measure theory and functional analysis that we require. To one extend or another, the results are standard. They are presented in a somewhat abstract fashion (i.e., without reference to the content regarding economics and finance). The reader already familiar with such things can refer to this section for notation. Others may be forced to slog through it, wondering how the machinery will be used.

In Section 2 we summarize the Krepsian approach to arbitrage as it bears on our analysis, and we present the skeleton structure of an apparent arbitrage, which we flesh out in the examples in the subsequent sections. Section 3 contains the first set of examples. There is no (explicit) uncertainty here; the examples are intended to allow the reader to dive into weak-\* convergence with as little overhead as possible.

Section 4 is the heart of the paper. In this section we provide examples in a countable space. The doubling strategy is presented in Example 4.2. In Section 5 we embed the setting of Section 4 in a dynamic securities market, complete with filtration, self-financing trading strategies, and an equivalent martingale measure. In an important sense, there is nothing new in this section: We show that the analysis of arbitrage opportunities in the static model in Section 4 carries over (with suitable changes in terminology) to the standard analysis of arbitrage opportunities in the dynamic setting (up to but not including doubling strategies, of course).

In Section 6 we reverse the roles of our dual pair of spaces in order to make contact with the Back and Pliska (1991) example. In Section 7 we show how our approach can be applied in the Black–Scholes setting. Although our approach here is suggestive, it is not entirely satisfactory owing to the fact that it does not take the underlying stochastic machinery seriously. Consequently, this section provides a jumping-off point for further research into applying the ideas we present here to continuous-time stochastic processes.

## 1. THE DUALITY OF FUNCTIONS AND MEASURES

Given a compact metric space  $X$ , let  $C(X)$  denote the space of continuous functions on  $X$ .<sup>5</sup> Equip  $C(X)$  with the sup norm topology (denoted  $\|\cdot\|_\infty$ ), where  $\|f\|_\infty = \sup \{|f(x)| : x \in X\}$ .  $C(X)$  is a Banach lattice. The topological dual space then can be identified with the space of finite signed measures  $M(X)$  on the Borel

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<sup>5</sup>Every compact metrizable space can be obtained as a quotient space from the Cantor set. See Appendix A for additional information.

$\sigma$ -algebra  $\mathcal{B}(X)$  generated from the Borel sets of  $X$ .  $M(X)$  is a Banach lattice under the total variation norm  $\|\mu\| = |\mu|(X) = \mu^+(X) + \mu^-(X)$  for  $\mu \in M(X)$ .<sup>6</sup>

We can equip  $M(X)$  with the weak-\* topology (denoted  $w^*$ ), in which case its topological dual space is  $C(X)$ . The pair of spaces  $\langle C(X), M(X) \rangle$  form a dual pair with the duality<sup>7</sup>

$$\langle f, \mu \rangle = \int_X f d\mu,$$

with  $f \in C(X)$  and  $\mu \in M(X)$ . Given  $\mu, \nu \in M(X)$ , the separation property of a duality implies

$$\mu = \nu \iff \langle f, \mu \rangle = \langle f, \nu \rangle \quad \forall f \in C(X). \quad (1.1)$$

Consider a sequence  $\{\mu_n\}_{n=1}^\infty \subset M(X)$  and an element  $\mu \in M(X)$ . Weak-\* convergence is characterized as follows:

$$\mu_n \xrightarrow{w^*} \mu \iff \langle f, \mu_n \rangle \rightarrow \langle f, \mu \rangle \quad \forall f \in C(X).$$

Strong convergence (i.e., convergence in norm) implies weak-\* convergence:

$$\|\mu_n - \mu\| \rightarrow 0 \implies \mu_n \xrightarrow{w^*} \mu.$$

A sequence  $\{\mu_n\}_{n=1}^\infty \subset M(X)$  is norm-bounded if  $\sup_{n \in \mathbb{N}} \|\mu_n\| < \infty$ . The unit ball in  $M(X)$  is weak-\* compact (Alaoglu's Theorem), and consequently every norm-bounded sequence of signed measures has weak-\* limit points in  $M(X)$ .

The unit ball in  $M(X)$  is weak-\* metrizable owing to the separability of  $C(X)$ . Let  $\{f_j\}_{j=1}^\infty$  be a dense subset of  $C(X)$ . Then weak-\* convergence need only be checked with respect to the elements of  $\{f_j\}_{j=1}^\infty$ . In particular,

$$\mu_n \xrightarrow{w^*} \mu \iff \langle f_j, \mu_n \rangle \rightarrow \langle f_j, \mu \rangle \quad \forall j \in \mathbb{N}.$$

Here is a metric compatible with the weak-\* topology:

$$d(\mu_1, \mu_2) = \sum_{j=1}^{\infty} 2^{-j} \wedge |\langle f_j, \mu_1 \rangle - \langle f_j, \mu_2 \rangle|.$$

Given a norm-bounded sequence  $\{\mu_n\}_{n=1}^\infty$ ,

$$\mu_n \xrightarrow{w^*} \mu \iff d(\mu_n, \mu) \rightarrow 0.$$

Given  $A \subset M(X)$ , the *annihilator* of  $A$  is defined by

$$A^\perp := \{f \in C(X) : \langle f, \mu \rangle = 0 \quad \forall \mu \in A\}. \quad (1.2)$$

<sup>6</sup>The lattice operations on  $M(X)$  are given by

$$\begin{aligned} \mu \vee \nu(A) &= \sup\{\mu(B) + \nu(B \setminus A) : B \in \mathcal{B}(X) \text{ and } B \subset A\} \\ \mu \wedge \nu(A) &= \inf\{\mu(B) + \nu(B \setminus A) : B \in \mathcal{B}(X) \text{ and } B \subset A\}. \end{aligned}$$

<sup>7</sup>The dual pair  $\langle C(X), M(X) \rangle$  is a Riesz pair (but not a symmetric Riesz pair). As such, a positive vector  $f \in C(X)^+$  is strictly positive, written  $f \gg 0$ , if  $f$  acts as a strictly positive linear functional on  $M(X)$  when considered as a member of  $M(X)^\sim$ , the order dual of  $M(X)$ . This requires  $f(x) > 0$  for all  $x \in X$ .

Theorem 5.96 in Aliprantis and Border (1999) asserts that if  $A$  is a subspace of  $M(X)$ , then the following are equivalent: (i)  $A^\perp = \{\mathbf{0}\}$ , (ii)  $A$  separates the points of  $C(X)$ , and (iii)  $A$  is weak-\* dense in  $M(X)$ .

Let  $\delta_x$  denote the point mass located at  $x \in X$ , where

$$\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \quad \forall A \in \mathcal{B}(X).$$

Note  $\|\delta_x\| = |\delta_x|(X) = \delta_x(X) = 1$ . Let  $\mathbf{1}_A$  denote the characteristic function for  $A \in \mathcal{B}(X)$  where

$$\mathbf{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

Let  $\mathbf{1} := \mathbf{1}_X$  and  $\mathbf{0} := \mathbf{1}_\emptyset$ .

Since  $X$  is compact metrizable, it is separable. Therefore, there is a countable dense set  $\{x_n\}_{n=1}^\infty \subset X$ . Since  $(\{\delta_{x_n}\}_{n=1}^\infty)^\perp = \{\mathbf{0}\}$ ,  $\{\delta_{x_n}\}_{n=1}^\infty$  is dense in  $M(X)$ . Thus finite linear combinations of  $\{\delta_{x_n}\}_{n=1}^\infty$  can be used to approximate any element of  $M(X)$  arbitrarily well. For example,  $\mathbb{N}$  is dense in  $\mathbb{N}_\infty$  (its one-point compactification). Therefore, the set  $\{\delta_x\}_{x \in \mathbb{N}}$  can be used to approximate  $\delta_\infty$  arbitrarily well. As another example, let  $\mathbb{Q}_{[0,1]}$  denote the rationals in  $[0, 1]$ . Since  $\mathbb{Q}_{[0,1]}$  is dense in  $[0, 1]$ , point masses on  $\mathbb{Q}_{[0,1]}$  can be used to approximate Lebesgue measure on  $[0, 1]$  arbitrarily well.

**Numeraire measure.** Fix a positive measure  $\lambda \in M(X)$ . By the Lebesgue decomposition, every signed measure  $\mu \in M(X)$  has a unique representation  $\mu = \mu^a + \mu^s$  where  $\mu^a \ll \lambda$  (i.e.,  $\mu^a$  is absolutely continuous with respect to  $\lambda$ ) and  $\mu^s \perp \lambda$  (i.e.,  $\mu^s$  and  $\lambda$  are mutually singular).<sup>8</sup> In fact,  $\lambda$  induces a decomposition of  $M(X)$  into the direct sum of projection bands:  $B_\lambda \oplus B_\lambda^d = M(X)$ , where

$$B_\lambda = \{\mu \in M(X) : \mu \ll \lambda\} \quad \text{and} \quad B_\lambda^d = \{\mu \in M(X) : \mu \perp \lambda\}.$$

The bands  $B_\lambda$  and  $B_\lambda^d$  are both linear subspaces of  $M(X)$  and they are disjoint complements: If  $\mu \in B_\lambda$  and  $\nu \in B_\lambda^d$ , then  $\mu \perp \nu$ .

Define  $L_1(\lambda) := L_1(X, \mathcal{B}(X), \lambda)$ . The norm of  $z \in L_1(\lambda)$  is  $\|z\|_1^\lambda = \int_X |z| d\lambda$ . Let  $\|\cdot\|_1^\lambda$  denote the norm topology. Note  $C(X) \subset L_1(\lambda)$ , since  $\|f\|_1^\lambda = \int_X |f| d\lambda < \infty$ . Consider the positive linear operator  $T_\lambda : B_\lambda \rightarrow L_1(\lambda)$ , where

$$T_\lambda(\mu) = d\mu/d\lambda. \tag{1.3}$$

Note  $\|T_\lambda(\mu)\|_1^\lambda = \|\mu\|$ . The inverse operator  $T_\lambda^{-1} : L_1(\lambda) \rightarrow B_\lambda$  is defined by  $T_\lambda^{-1}(z)(B) = \int_B z d\lambda$  for all  $B \in \mathcal{B}(X)$ . Since  $T_\lambda$  is a lattice isometry, the Banach lattices  $B_\lambda$  and  $L_1(\lambda)$  are identical from the point of view of Riesz spaces.

<sup>8</sup>In a Riesz space each pair of vectors has a supremum and an infimum:  $x \vee y = \sup\{x, y\}$  and  $x \wedge y = \inf\{x, y\}$ . In particular,  $x^+ = x \vee 0$ ,  $x^- = (-x) \vee 0$ ; in addition,  $|x| = x^+ + x^-$ . Vectors  $x$  and  $y$  are *disjoint* or *orthogonal*, written  $x \perp y$ , if  $|x| \wedge |y| = 0$ . From Lemma 9.56 in Aliprantis and Border (1999) we have the following: Given  $\mu, \nu \in M(X)$ ,  $\mu \perp \nu$  if and only if there exists some  $A \in \mathcal{B}(X)$  such that  $|\mu|(A) = |\nu|(A^c) = 0$ .

The *support* of a positive measure  $\lambda$  (if it exists<sup>9</sup>) is a closed set (denoted  $\text{supp } \lambda$ ) satisfying (i)  $\lambda((\text{supp } \lambda)^c) = 0$  and (ii) if  $A$  is open and  $A \cap \text{supp } \lambda \neq \emptyset$ , then  $\lambda(A \cap \text{supp } \lambda) > 0$ . For our purposes, what is important is this: If  $\text{supp } \lambda = X$ , then  $B_\lambda^\perp = \{\mathbf{0}\}$  and therefore  $B_\lambda$  is dense in  $M(X)$ . If  $B_\lambda$  is dense in  $M(X)$ , then any element of  $M(X)$  can be approximated arbitrarily well by elements in  $B_\lambda$ , all of which have densities with respect to  $\lambda$ . Consequently, we will call  $\lambda$  a *numeraire measure* if  $0 \leq \lambda \in M(X)$  and  $\text{supp } \lambda = X$ .

*Natural measure.* Fix a positive measure  $\varphi \in B_\lambda$  for which  $\lambda \ll \varphi$ . In other words,  $\varphi$  is equivalent to  $\lambda$ . Since  $\varphi$  and  $\lambda$  are equivalent,  $B_\varphi = B_\lambda$ . Then  $\xi := T_\lambda(\varphi) = d\varphi/d\lambda$  is the density of the payout  $\varphi$  in terms of the numeraire measure. Define  $\pi := T_\varphi(\lambda) = d\lambda/d\varphi = \xi^{-1}$ . Note  $T_\lambda(\mu)T_\varphi(\lambda) = (d\mu/d\lambda)(d\lambda/d\varphi) = d\mu/d\varphi = T_\varphi(\mu)$ . Then for  $\mu \in B_\lambda$ ,

$$\langle \mathbf{1}, \mu \rangle = \int_X z \pi d\varphi,$$

where  $z = T_\lambda(\mu)$ . If  $\varphi(X) = 1$ , then  $\int_X z \pi d\varphi = E^\varphi[z \pi]$ . If, in addition,  $\varphi$  is the *natural measure* (also known as the *physical measure*), then we say  $\pi$  is the state-price deflator.

*Reference measure.* Define  $L_1(\eta) := L_1(X, \mathcal{B}(X), \eta)$  where  $\eta$  is a positive  $\sigma$ -finite Borel measure  $\eta$  such that  $\lambda \ll \eta$ . (Note  $\eta(X) < \infty \implies \eta \in M(X)$ .) We refer to  $\eta$  as the *reference measure*. (In our examples,  $\eta$  is either the counting measure or Lebesgue measure.) We refer to  $G := d\lambda/d\eta = T_\eta(\lambda) \in L_1(\eta)$  as the *pricing function*. Note  $\lambda = T_\eta^{-1}(G) \in M(X)$ . In addition,  $z \in L_1(\lambda) \iff zG \in L_1(\eta)$  and  $\int_X z d\lambda = \int_X zG d\eta$ . Given  $\eta$ , choosing  $G \in L_1(\eta)$  is equivalent to choosing  $\lambda \in M(X)$ .

Finally, let  $F = d\varphi/d\eta$ . Then

$$E^\varphi[z \pi] = \int_X z \pi F d\eta = \int_X \left( \frac{d\mu}{d\lambda} \right) \left( \frac{d\lambda}{d\varphi} \right) \left( \frac{d\varphi}{d\eta} \right) d\eta = \int_X d\mu,$$

where  $\mu, \lambda, \varphi$ , and  $\eta$  are all Borel measures on  $\mathcal{B}(X)$  and (with the possible exception of  $\eta$ ) all are in  $M(X)$ .

**Convergence in  $L_1(\lambda)$  and uniform integrability.** Given a sequence  $\{z_n\}_{n=1}^\infty$  where  $z_n \in L_1(\lambda)$  and  $z \in L_1(\lambda)$ . Consider the following modes of convergence. First, convergence in the  $L_1$  norm:

$$z_n \xrightarrow{\|\cdot\|_1^\lambda} z \iff \|z_n - z\|_1^\lambda \rightarrow 0.$$

We can restate the norm convergence of functions in terms of the norm convergence of measures: Given a sequence  $\{\mu_n\}_{n \in \mathbb{N}} \subset B_\lambda$  and  $\mu \in B_\lambda$ ,

$$\|\mu_n - \mu\| \rightarrow 0 \iff \|T_\lambda(\mu_n - \mu)\|_1^\lambda \rightarrow 0.$$

Second, convergence  $\lambda$ -almost everywhere:

$$z_n \xrightarrow{\lambda\text{-a.e.}} z \iff z_n(x) - z(x) \rightarrow 0 \quad \forall x \in X \setminus E \quad \text{where } \lambda(E) = 0.$$

<sup>9</sup>Given the separability of  $X$ , the support of  $0 \leq \lambda \in M(X)$  exists.

Third, convergence in  $\lambda$ -measure:

$$z_n \xrightarrow{\lambda} z \iff \lambda(\{x \in X : |z_n(x) - z(x)| > \varepsilon\}) \rightarrow 0 \quad \forall \varepsilon > 0.$$

Relations among the modes of convergence:

$$z_n \xrightarrow{\lambda\text{-a.e.}} z \implies z_n \xrightarrow{\lambda} z$$

and

$$z_n \xrightarrow{\|\cdot\|_1^\lambda} z \implies (i) z_n \xrightarrow{\lambda} z \text{ and } (ii) z \text{ is the unique } \lambda\text{-a.e. limit point.}$$

**Theorem 1.** Fix  $0 \leq \lambda \in M(X)$ . Given  $\{\mu_n\}_{n=1}^\infty \subset B_\lambda$  and  $\mu = \mu^a + \mu^s$ , where  $\mu^a \in B_\lambda$  and  $\mu^s \in B_\lambda^d$ . Then  $\mu_n \xrightarrow{w^*} \mu \implies T_\lambda(\mu_n) \xrightarrow{\lambda} T_\lambda(\mu^a)$ .

*Proof.* Christian will supply proof. □

The sequence  $\{z_n\}_{n \in \mathbb{N}} \subset L_1(\lambda)$  is uniformly integrable (UI) if

$$\lim_{\alpha \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{|z_n| \geq \alpha} |z_n| d\lambda = 0.$$

**Theorem 2.** Fix  $0 \leq \lambda \in M(X)$ . Given the sequence  $\{\mu_n\}_{n \in \mathbb{N}} \subset B_\lambda$ , the sequence  $\{T_\lambda(\mu_n)\}_{n \in \mathbb{N}}$  is UI if and only if

$$\sup_{n \in \mathbb{N}} \|\mu_n\| < \infty \quad \text{and} \quad \lim_{\lambda(A) \rightarrow 0} \sup_{n \in \mathbb{N}} |\mu_n|(A) = 0.$$

Doob (1994) describes the second condition as *uniform absolute continuity* of the sequence of measures  $\{\mu_n\}_{n \in \mathbb{N}}$ .

*Proof.* See Doob (1994, p. 94). □

**Theorem 3.** Fix  $0 \leq \lambda \in M(X)$ . Given  $\{z_n\}_{n=1}^\infty \subset L_1(\lambda)$ . If  $z_n \xrightarrow{\lambda} z$ , then  $\|z_n - z\|_1^\lambda \rightarrow 0$  if and only if  $\{z_n\}_{n=1}^\infty$  is UI.

*Proof.* See Doob (1994, p. 95). □

**Theorem 4** (Dunford–Pettis). A subset of  $L_1(\lambda)$  is UI if and only if it is relatively weakly compact.

The following theorem is implied by the Dunford–Pettis Theorem.

**Theorem 5.** Fix  $0 \leq \lambda \in M(X)$ . Given  $S := \{\mu_n\}_{n=1}^\infty \subset B_\lambda$ . Let  $\bar{S}$  denote the weak-\* closure of  $S$  in  $M(X)$ . Then  $\{T_\lambda(\mu_n)\}_{n=1}^\infty$  is UI if and only if  $\bar{S} \subset B_\lambda$ .

*Proof.* To be supplied. □

**Corollary 1.** Fix  $0 \leq \lambda \in M(X)$ . Given  $\{\mu_n\}_{n=1}^\infty \subset B_\lambda$  and  $\mu \in B_\lambda$ . Then  $\mu_n \xrightarrow{w^*} \mu \iff \mu_n \xrightarrow{\|\cdot\|_1} \mu$ .

## 2. ARBITRAGE

Our treatment of arbitrage follows Kreps (1981) and Clark (1993, 2000).<sup>10</sup> Fix a topological space of payouts  $(S, \tau)$  and a cone  $K$ . In this paper,  $S$  is a Banach lattice and  $K = S^+$ , the positive cone of  $S$ . Let  $S^{++} = S^+ \setminus \{0\}$  denote the strictly positive cone (i.e., the positive cone with the origin deleted). The set of marketed securities is  $\mathbb{M}_0$ . The space of all marketed claims  $\mathbb{M} = \mathbf{sp}(\mathbb{M}_0)$ , the linear span of  $\mathbb{M}_0$ , is a linear subspace of  $S$  and the prices of marketed contingent claims are given by a linear functional  $V : \mathbb{M} \rightarrow \mathbb{R}$ .

Define the feasible set  $F := \{q \in \mathbb{M} : V(q) \leq 0\}$ . An arbitrage is an element of  $S^{++} \cap F$ . We can characterize the absence of arbitrage opportunities in terms of the set of zero-cost marketed claims  $F_0 := \{q \in \mathbb{M} : V(q) = 0\}$ :

$$S^{++} \cap F_0 = \emptyset. \quad (\text{NA})$$

It can be shown that (NA) holds if and only if  $V$  is strictly positive. A strictly positive functional has the following property:  $q \in S^{++} \implies V(q) > 0$ .

Let  $\overline{F}$  denote the closure of  $F$ . An approximate arbitrage is an element of  $S^{++} \cap \overline{F}$ . Let  $\overline{F_0}$  denote the closure of  $F_0$ . We can characterize the absence of approximate arbitrage opportunities in terms of  $\overline{F_0}$ , the closure of  $F_0$ :

$$S^{++} \cap \overline{F_0} = \emptyset. \quad (\text{NAA})$$

It can be shown that (NAA) holds if and only if  $V$  has a strictly positive and continuous extension to  $S$ .<sup>11</sup> A continuous linear functional is a member of the topological dual space  $(S, \tau)^*$ .

*Spaces of payouts.* Given  $0 \leq \lambda \in M(X)$  where  $\text{supp } \lambda = X$ . The following are equivalent:

- (1)  $z \mapsto \int_X z d\lambda$  is a valuation operator on  $(L_1(\lambda), \|\cdot\|_1^\lambda)$ .
- (2)  $\mu \mapsto \int_X d\mu = \langle \mathbf{1}, \mu \rangle$  is a valuation operator on  $(B_\lambda, \|\cdot\|)$ .
- (3)  $\mu \mapsto \int_X d\mu = \langle \mathbf{1}, \mu \rangle$  is a valuation operator on  $(M(X), w^*)$ .

**Anatomy of an apparent arbitrage.** Let the space of payouts be  $(M(X), w^*)$  and let the valuation operator be given by  $\mu \mapsto \langle \mathbf{1}, \mu \rangle$ . Fix a positive measure  $\lambda \in M(X)$  for which  $\text{supp } \lambda = X$ . Take as given a sequence  $\{\mu_n\}_{n=1}^\infty \subset B_\lambda$  where  $\mu_n(X) = 0$  for all  $n \in \mathbb{N}$  and for which  $\mu_n \xrightarrow{w^*} \mu = \mu^a + \mu^s$ , where  $\mu^a \in B_\lambda$  and  $\mu^s \in B_\lambda^d$ . By Theorem 1,  $T_\lambda(\mu_n) \xrightarrow{\lambda} T_\lambda(\mu^a)$ . It follows from weak-\* convergence that  $\mu(X) = 0$  and thus  $\int_X T_\lambda(\mu^a) d\lambda = -\mu(X)$ . If  $\mu^s(X) < 0$  we have an apparent arbitrage opportunity, while if  $\mu^s(X) > 0$  we have an apparent suicide strategy.<sup>12</sup>

<sup>10</sup>Which see for omitted details.

<sup>11</sup>As noted in the introduction, Kreps (1981) uses the setting we have adopted as an example. In particular, he shows that *viability* is equivalent to *no free lunches* (NFL) in our setting. In addition, NAA is equivalent to NFL in our setting.

<sup>12</sup>Note, if  $\mu^s \neq 0$  (even if  $\mu^s(X) = 0$ ), then  $\|\mu_n - \mu\| \not\rightarrow 0$ .

3. INTERVALS ON THE REAL LINE:  $X = [0, 1]$  AND  $X = [0, \infty]$ 

In this section we illustrate weak-\* convergence and apparent arbitrages in the spaces  $X = [0, 1]$  and  $X = [0, \infty]$ . In all cases, the reference measure  $\eta$  is Lebesgue measure, denoted  $\text{Leb}$ .

**Example 3.1.** This example shows that a sequence of ‘lump-sum’ payouts can converge to a ‘lump-sum’ payout. Let  $X = [0, 1]$ . Given a sequence of distinct points  $\{x_n\}_{n=1}^\infty \subset [0, 1] \setminus \{x\}$  such that  $x_n \rightarrow x \in [0, 1]$ . The sequence of point masses  $\{\delta_{x_n}\}_{n=1}^\infty$  does not converge in norm, since  $\|\delta_{x_n} - \delta_x\| = 2$  for all  $n \in \mathbb{N}$ . Nevertheless, the sequence does converge in the weak-\* topology:

$$\langle f, \delta_{x_n} \rangle = f(x_n) \xrightarrow{n} f(x) = \langle f, \delta_x \rangle \quad \forall f \in C([0, 1]).$$

**Example 3.2.** This example illustrates how ‘lump-sum’ payouts can be used to approximate a ‘flow’ payout. Let  $X = [0, 1]$  and let  $\lambda = \text{Leb}$ . For each  $n \in \mathbb{N}$ , partition  $[0, 1]$  into  $n$  intervals of equal length. Define

$$\hat{\lambda}_n := \frac{1}{n} \sum_{i=1}^n \delta_{\frac{2i-1}{2n}}.$$

Note  $\hat{\lambda}_n \xrightarrow{w^*} \lambda$  since

$$\int_{[0,1]} f d\hat{\lambda}_n = \frac{1}{n} \sum_{i=1}^n f\left(\frac{2i-1}{2n}\right) \xrightarrow{n} \int_0^1 f(x) dx = \langle f, \lambda \rangle \quad \forall f \in C([0, 1]).$$

**Example 3.3.**

**Example 3.4.** This example illustrates an apparent arbitrage. Let  $X = [0, 1]$  and let  $\lambda = \text{Leb}$ . Let  $\mu = \lambda - \delta_0$ , so that  $\mu(X) = 0$  and  $\|\mu\| = 2$ . In addition,  $\mu^a = \lambda$ ,  $\mu^s = -\delta_0$ ,  $z = d\mu^a/d\lambda = \mathbf{1}$  ( $\lambda$ -a.e.), and  $\int_X z d\lambda = 1$ .

Define  $\mu_n$  via  $\mu_n(A) = \int_A z_n d\lambda$  for all  $A \in \mathcal{B}(X)$ , where

$$z_n(x) := \begin{cases} 1 - 2^{n-1} & x \in [0, 2^{1-n}] \\ 1 & x \in (2^{1-n}, 1]. \end{cases}$$

Note  $\mu_n \xrightarrow{w^*} \mu$  and  $z_n \xrightarrow{\lambda\text{-a.e.}} z$ . However,  $\mu_n(X) = \int_X z_n d\lambda = 0$  for all  $n \in \mathbb{N}$ . Since  $\int_X z d\lambda = 1$ , there is an apparent arbitrage.

**Example 3.5.** (Miller–Modigliani). This example is adapted from the ‘zero’-dividend example in Gilles and LeRoy (1997).

Consider a firm that starts with one unit of capital that generates earnings at rate  $r > 0$ . Suppose the firm pays out a constant fraction  $\gamma \in (0, 1)$  of earnings as dividends. Let  $z_\gamma(t)$  denote the rate of flow of dividends at time  $t$  and let  $s(t)$  denote firm’s stock of capital at time  $t$ . Then  $z_\gamma(t) = \gamma r s(t)$ . The firm acquires additional capital with its retained earnings, and so its capital evolves according to the ordinary differential equation (ODE)  $s'(t) = (1 - \gamma) r s(t)$  subject to  $s(0) = 1$ . The solution to the ODE is  $s(t) = e^{(1-\gamma)rt}$  and therefore  $z_\gamma(t) = \gamma r e^{(1-\gamma)rt}$ .

Let  $X = [0, \infty]$  and let  $\eta = \text{Leb}$ . Thus  $z_\gamma$  is the density of the payout (i.e., the dividend) with respect to Lebesgue measure. Let  $G(t) = e^{-rt}$  be the pricing function, so the numeraire measure is characterized by  $\lambda(dt) = G(t)\eta(dt) = G(t)dt$ . Note  $\lambda(X) = \int_0^\infty e^{-rt} dt = 1/r$ . Define  $\mu_\gamma = T_\lambda^{-1}(z_\gamma)$ . Note  $w_\gamma(t) := \gamma r e^{-\gamma r t}$  is the density of  $\mu_\gamma$  with respect to Lebesgue measure. The value of the firm is independent of the dividend–payout ratio:

$$\mu_\gamma(X) = \int_0^\infty \mu_\gamma(dt) = \int_0^\infty \gamma r e^{-\gamma r t} dt = 1.$$

Consider what happens when the dividend–payout ratio goes to zero. First, note that all of the firm’s value is attributable to dividends in the tail as the dividend–payout ratio goes to zero:

$$\lim_{\gamma \downarrow 0} \mu_\gamma([T, \infty]) = 1 \quad \text{for every finite } T.$$

Given a sequence  $\{\gamma_n\}_{n=1}^\infty \subset (0, 1)$  where  $\gamma_n \rightarrow 0$ , we show that  $\mu_{\gamma_n} \xrightarrow{w^*} \delta_\infty$ . Note

$$\langle f, \mu_\gamma \rangle = \int_0^\infty f(t) d\mu_\gamma(dt) = \int_0^\infty f(t) w_\gamma(t) dt.$$

Thus,

$$\langle f, \mu_{\gamma_n} \rangle \xrightarrow{n} f(\infty) = \langle f, \delta_\infty \rangle \quad \forall f \in C([0, \infty]),$$

where  $f(\infty) := \lim_{t \rightarrow \infty} f(t)$ . Note that buying shares in the ‘zero’-dividend firm is an apparent suicide strategy.

*Convergence in distribution.* As an aside, note that  $w_\gamma$  has the form of the probability density function of the exponential distribution with location equal to zero and scale equal to  $(\gamma r)^{-1}$ . The corresponding cumulative distribution function is given by  $F_\gamma(t) = 1 - e^{-\gamma r t}$ . Note,  $\lim_{\gamma \downarrow 0} F_\gamma(t) = 0$  for all  $t \in [0, \infty)$ . The sequence of cumulative distribution functions  $\{F_{\gamma_n}\}_{n=1}^\infty$  converges in distribution to

$$F_0(t) = \begin{cases} 0 & t < \infty \\ 1 & t = \infty. \end{cases}$$

#### 4. COUNTABLE STATE SPACE: $X = \mathbb{N}_\infty$

In this section, let  $X = \mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$ .<sup>13</sup>  $\mathbb{N}_\infty$  is the one-point compactification of  $\mathbb{N}$ .<sup>14</sup> Let  $e_n \in \mathbb{N}_\infty$  denote the  $n$ -th unit coordinate vector; i.e., the sequence whose  $n$ -th term is one and every other term is zero and let  $\tilde{e}_n := \sum_{i=n+1}^\infty e_i$ . Note  $\mathbf{1} = \mathbf{1}_{\mathbb{N}_\infty} = \mathbf{1}_{\mathbb{N}} + e_\infty = \tilde{e}_0 + e_\infty$ .

<sup>13</sup>In Appendix A we show how to obtain  $\mathbb{N}_\infty$  as a quotient space of the Cantor space of infinite sequences of zeros and ones.

<sup>14</sup>The open sets of  $\mathbb{N}_\infty$  are the open sets of  $\mathbb{N}$  in the discrete topology and sets of the form  $A \cup \{\infty\}$  where  $A$  is an open subset of  $\mathbb{N}$  and  $\mathbb{N} \setminus A$  is compact. Thus  $A$  is an infinite set.

Given a sequence  $\{f(i)\}_{i=1}^{\infty}$ , define  $\text{Lim}[f] := \lim_{i \rightarrow \infty} f(i)$ . Note that  $f \in C(\mathbb{N}_{\infty}) \iff \text{Lim}[f]$  exists and is finite and  $f(\infty) = \text{Lim}[f]$ . In addition, note that  $\mu \in M(\mathbb{N}_{\infty}) \iff \mu = \sum_{i=1}^{\infty} m_i \delta_i + m_{\infty} \delta_{\infty}$ , subject to

$$\|\mu\| = |\mu|(\mathbb{N}_{\infty}) = \sum_{i=1}^{\infty} |m_i| + |m_{\infty}| < \infty.$$

Thus,  $\mu(\{i\}) = m_i$  for all  $i \in \mathbb{N}_{\infty}$ . The duality for  $\langle C(\mathbb{N}_{\infty}), M(\mathbb{N}_{\infty}) \rangle$  is<sup>15,16</sup>

$$\langle f, \mu \rangle = \int_X f d\mu = \sum_{i=1}^{\infty} f(i) m_i + \text{Lim}[f] m_{\infty}.$$

Let  $\lambda := \sum_{i=1}^{\infty} \beta_i \delta_i$ , where  $\beta_n > 0$  for  $n \in \mathbb{N}$  and  $\bar{\beta} := \sum_{i=1}^{\infty} \beta_i < \infty$ .<sup>17</sup> Define  $\tilde{\beta}_n := \sum_{i=n+1}^{\infty} \beta_i$ . Note  $\lambda(\mathbb{N}_{\infty}) = \bar{\beta} < \infty$  and  $\lambda(\{\infty\}) = 0$ . The support of  $\lambda$  is  $\mathbb{N}_{\infty}$ .<sup>18</sup>

For any  $\mu \in M(\mathbb{N}_{\infty})$ , we have  $\mu = \mu^a + \mu^s$  where  $\mu^a = \sum_{i=1}^{\infty} m_i \delta_i \ll \lambda$  and  $\mu^s = m_{\infty} \delta_{\infty} \perp \lambda$ . For  $\mu \in B_{\lambda}$ ,  $T_{\lambda}(\mu) = d\mu/d\lambda = \sum_{i=1}^{\infty} (m_i/\beta_i) e_i \in L_1(\lambda)$ .<sup>19</sup> Given  $z = \sum_{i=1}^{\infty} z(i) e_i \in L_1(\lambda)$ ,  $T_{\lambda}^{-1}(z) = \sum_{i=1}^{\infty} z(i) \beta_i \delta_i$ .<sup>20</sup>

**Example 4.1.** We have  $\delta_n \xrightarrow{w^*} \delta_{\infty}$  since

$$\langle f, \delta_n \rangle = f(n) \xrightarrow{n} \text{Lim}[f] = \langle f, \delta_{\infty} \rangle \quad \forall f \in C(\mathbb{N}_{\infty}).$$

The density of  $\delta_n$  with respect to  $\lambda$  is  $T_{\lambda}(\delta_n) = \beta_n^{-1} e_n$  for  $n \in \mathbb{N}$ . Note  $\lim_{n \rightarrow \infty} \beta_n^{-1} = \infty$ . Since  $\delta_{\infty} \in B_{\lambda}^d$ ,  $\{T_{\lambda}(\delta_n)\}_{n=1}^{\infty}$  is not uniformly integrable.

*Interpretation.* The very-long discount (VLD) bond is the weak-\* limit of a sequence of zero-coupon bonds. (This example is adapted from Gilles and LeRoy (1997).)

We interpret  $\beta_n$  as the discount factor:  $\beta_n = (1+r)^{-n}$ , where  $r > 0$  is the interest rate.<sup>21</sup> The payout to a zero-coupon bond that pays one unit when it matures at time  $n$  is  $e_n$  and its value is  $\beta_n$ . The payout to a zero-coupon bond that pays  $(1+r)^n$  when it matures at time  $n$  is  $\beta_n^{-1} e_n$  and its value is 1. We can identify the payout to this bond with the measure  $\delta_n = T_{\lambda}^{-1}(\beta_n^{-1} e_n)$ . The payout to the VLD bond is  $\delta_{\infty}$  and its value is  $\delta_{\infty}(\mathbb{N}_{\infty}) = 1$ .

The payout to the VLD bond can be replicated sequentially as follows. Invest \$1 at time zero in one-period debt and rollover the investment each period. The reverse transaction (a Ponzi scheme of sorts: borrow \$1 at time zero and rollover the outstanding debt each period) is an apparent arbitrage.

<sup>15</sup> $C(\mathbb{N}_{\infty})$  is also known as  $c$  and  $M(\mathbb{N}_{\infty})$  is also known as  $\ell_1 \oplus \mathbb{R}$ .

<sup>16</sup>The duality for  $\langle B_b(\mathbb{N}_{\infty}), M(\mathbb{N}_{\infty}) \rangle$  is  $\langle f, \mu \rangle = \sum_{i=1}^{\infty} f(i) m_i + f(\infty) m_{\infty}$ , where  $f \in B_b(\mathbb{N}_{\infty})$  if  $\|f\|_{\infty} < \infty$ . Even if  $\text{Lim}[f]$  exists, we need not have  $f(\infty) = \text{Lim}[f]$ .

<sup>17</sup>The reference measure is the counting measure  $\eta = \sum_{i=1}^{\infty} \delta_i + \delta_{\infty}$  and the pricing function is  $G = d\lambda/d\eta = \sum_{i=1}^{\infty} \beta_i e_i$ .

<sup>18</sup> $\mathbb{N}_{\infty}$  is closed, its complement  $\emptyset$  has  $\lambda$ -measure zero, and  $\{\infty\}$  is not open (so the  $\lambda$ -measure of  $\mathbb{N}_{\infty} \cap \{\infty\} = \{\infty\}$  need not be positive).

<sup>19</sup> $T_{\lambda}(\mu)(\infty)$  is arbitrary since  $\lambda(\{\infty\}) = 0$ . We have set it to zero for convenience.

<sup>20</sup> $T_{\lambda}^{-1}(z)$  does not depend on  $z(\infty)$ , which we have set to zero for convenience.

<sup>21</sup>Note  $\bar{\beta} = 1/r$ . If  $r = 0$ , then  $\lambda(\mathbb{N}_{\infty}) = \infty$  and  $\lambda$  would be  $\sigma$ -finite but not finite.

**Example 4.2.** Define

$$\tilde{\delta}_n := \tilde{\beta}_n^{-1} \sum_{i=n+1}^{\infty} \beta_i \delta_i. \quad (4.1)$$

Note  $\|\tilde{\delta}_n\| = \tilde{\delta}_n(\mathbb{N}_\infty) = \tilde{\beta}_n^{-1} \sum_{i=n+1}^{\infty} \beta_i = 1$ . We have  $\tilde{\delta}_n \xrightarrow{w^*} \delta_\infty$  since

$$\langle f, \tilde{\delta}_n \rangle = \tilde{\beta}_n^{-1} \sum_{i=n+1}^{\infty} \beta_i f(i) \xrightarrow{n} \text{Lim}[f] = \langle f, \delta_\infty \rangle \quad \forall f \in C(\mathbb{N}_\infty).$$

The density of  $\tilde{\delta}_n$  with respect to  $\lambda$  is

$$T_\lambda(\tilde{\delta}_n) = \tilde{\beta}_n^{-1} \tilde{e}_n. \quad (4.2)$$

Note  $\lim_{n \rightarrow \infty} \tilde{\beta}_n^{-1} = \infty$ . Since  $\delta_\infty \in B_\lambda^d$ ,  $\{T_\lambda(\tilde{\delta}_n)\}_{n=1}^\infty$  is not uniformly integrable. Also note  $T_\lambda(\tilde{\delta}_n) \xrightarrow{\lambda\text{-a.e.}} \mathbf{0}$ .

*Interpretation.* Let the state of the world be characterized by the first occurrence of RED on a roulette wheel. The suicide strategy is the weak-\* limit of a sequence of normalized fair bets on BLACK.

We interpret  $\beta_n$  as the probability that the first RED occurs on the  $n$ -th spin. We assume  $\bar{\beta} = 1$ . The price of a fair bet that pays one unit if the first RED occurs on the  $n$ -th spin equals the probability,  $\beta_n$ . The payout of this bet is  $e_n$ . It is convenient to normalize fair bets so that their cost is \$1. Thus the payout to a normalized fair bet that the first RED occurs on the  $n$ -th spin is  $\beta_n^{-1} e_n$ . (For example, if  $\beta_n = 2^{-n}$ , then  $\beta_n^{-1} = 2^n$ .)

The payout to a normalized fair bet that the first  $n$  spins are all BLACK is  $\tilde{\beta}_n^{-1} \tilde{e}_n$ . (For example, if  $\beta_n = 2^{-n}$ , then  $\tilde{\beta}_n^{-1} = 2^n$ .) We can identify this payout with the measure  $\tilde{\delta}_n = T_\lambda^{-1}(\tilde{\beta}_n^{-1} \tilde{e}_n)$ . The sequence of payouts  $\{\tilde{\delta}_n\}_{n=1}^\infty$  converges to  $\delta_\infty$ , which is the payout to a normalized fair bet that the first RED never occurs. This payout does not have a density with respect to  $\lambda$ ; nevertheless, the payout can be approximated arbitrarily well with payouts that do have densities. The bet can be executed sequentially by first betting \$1 on BLACK and continuing to bet any and all winnings on BLACK until RED occurs.

**The doubling strategy.** The doubling strategy involves betting on RED and doubling the bet each time BLACK occurs. The first bet of \$1 is financed by borrowing, as are all subsequent required bets. When RED occurs, the loans are repaid, after which \$1 remains.

Formally, we can model the doubling strategy as follows: Define  $\mu_n := \lambda - \tilde{\delta}_n$ . Note  $\mu_n(\mathbb{N}_\infty) = 0$ . The payout to the doubling strategy (in terms of the numeraire) after the  $n$ -th spin is  $T_\lambda(\mu_n) = \mathbf{1}_\mathbb{N} - \tilde{\beta}_n^{-1} \tilde{e}_n$ . For example, if  $\beta_n = 2^{-n}$ , then  $T_\lambda(\mu_n) = \sum_{i=1}^n e_i + (1 - 2^n) \tilde{e}_n$ . Finally, we have  $\mu_n \xrightarrow{w^*} \lambda - \delta_\infty \notin M(X)^{++}$ .

**Convergence in distribution of the suicide strategy.** Given a sequence of random variables defined on  $\mathbb{N}_\infty$ , we can examine the corresponding sequence of image measures on the real line and see to what it converges in distribution. We show that the suicide strategy converges in distribution to zero.

The distribution function for  $T_\lambda(\tilde{\delta}_n) = \beta_n^{-1} \tilde{e}_n$  is

$$F_n(x) := \lambda(\{i : T_\lambda(\tilde{\delta}_n)(i) \leq x\}) = \begin{cases} 0 & x < 0 \\ 1 - \tilde{\beta}_n & 0 \leq x < \tilde{\beta}_n^{-1} \\ 1 & \tilde{\beta}_n^{-1} \leq x. \end{cases}$$

Note,

$$\int_{\mathbb{R}} f dF_n = (1 - \tilde{\beta}_n) f(0) + \tilde{\beta}_n f(\tilde{\beta}_n^{-1}) \quad \forall f \in C_b(\mathbb{R}),$$

where  $C_b(\mathbb{R})$  is the space of continuous bounded functions on  $\mathbb{R}$ . Also note,

$$\lim_{n \rightarrow \infty} \tilde{\beta}_n = 0.$$

Therefore,

$$\int_{\mathbb{R}} f dF_n \rightarrow f(0) = \int_{\mathbb{R}} f dF \quad \forall f \in C_b(\mathbb{R}),$$

where

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

is the distribution for a random variable that is identically zero. In other words, we find that the sequence of random variables converges in distribution to the constant zero. This does not contradict our earlier finding that the suicide strategy does not weak-\* converge to zero. Although both sequences involve weak convergence in the generic sense, the spaces of measures (and of test functions) are different. On the one hand we have  $\{\tilde{\delta}_n\}_{n=1}^\infty$ , a sequence of measures on  $\mathbb{N}_\infty$ , while on the other hand we have  $\{F_n\}_{n=1}^\infty$ , a sequence of distribution functions for Lebesgue–Stieltjes measures on the real line.

**Arbitrage and approximate arbitrage.** Here we examine the conditions for no arbitrage and no approximate arbitrage. The marketed securities are the Arrow–Debreu securities (one for each finite state) and a bond that pays one unit in every finite state.

The payout to the  $n$ -th A–D security is  $e_n$  and the payout to the bond is  $\mathbf{1}_\mathbb{N} = \sum_{i=1}^\infty e_i$ . The set of marketed payouts is  $\mathbb{M}_0 := \{\mathbf{1}_\mathbb{N}\} \cup \{e_n\}_{n=1}^\infty$ . The space of marketed payouts is  $\mathbb{M} := \mathbf{sp}(\mathbb{M}_0)$ . Note  $z \in \mathbb{M}$  if and only if  $z = a_0 \mathbf{1}_\mathbb{N} + \sum_{i=1}^\infty a_i e_i$  where the sequence  $\{a_i\}_{i=1}^\infty$  has finite support. The cost of  $z \in \mathbb{M}$  is  $V(z) = a_0 V_0(\mathbf{1}_\mathbb{N}) + \sum_{i=1}^\infty a_i V_0(e_i)$ , where  $V_0(\mathbf{1}_\mathbb{N})$  and  $V_0(e_n)$  are the given prices of the marketed securities. Let  $V_0(e_n) = \beta_n$  and  $V_0(\mathbf{1}_\mathbb{N}) = 1$ .<sup>22</sup> Thus  $V(z) = a_0 + \sum_{i=1}^\infty a_i \beta_i$ .

<sup>22</sup>See Werner (1997) for a related example. In Werner,  $\beta_i = (2i(i+1))^{-1}$  and  $\bar{\beta} = 1/2$ .

*Arbitrage.* There are no arbitrage opportunities if and only if  $V$  is a strictly positive linear functional. The following conditions are necessary and sufficient for the absence of arbitrage:

$$\beta_n > 0 \quad \forall n \in \mathbb{N} \quad \text{and} \quad \bar{\beta} \leq 1. \quad (4.3)$$

We address necessity first. If  $\beta_n \leq 0$ , then  $z = \mathbf{e}_n - \beta_n \mathbf{1}_{\mathbb{N}}$  is an arbitrage since  $V(z) = 0$  and  $z$  is nonnegative and not zero. On the other hand, if  $\bar{\beta} > 1$ , then there is some finite  $n$  for which  $\sum_{i=1}^n \beta_i = b > 1$ . In this case,

$$z = b \mathbf{1}_{\mathbb{N}} - \sum_{i=1}^n \mathbf{e}_i = \sum_{i=1}^n (b-1) \mathbf{e}_i + b \sum_{i=n+1}^{\infty} \mathbf{e}_i$$

is an arbitrage since  $V(z) = 0$  and  $z$  is nonnegative and not zero.

We now show that conditions (4.3) are sufficient to guarantee the absence of arbitrage by displaying the requisite linear functional. For  $z \in C(\mathbb{N}_{\infty}) \supset \mathbb{M}$ , we can express  $V$  as the strictly positive linear functional<sup>23</sup>

$$V(z) = \int_X z d\lambda + \text{Lim}[z] (1 - \bar{\beta}). \quad (4.4)$$

Consequently, there are no arbitrage opportunities in the marketed subspace  $\mathbb{M}$ . This is true even though  $\sum_{i=1}^{\infty} V(\mathbf{e}_i) \neq V(\sum_{i=1}^{\infty} \mathbf{e}_i)$ .

*Approximate arbitrage.* There are no approximate arbitrage opportunities if and only if  $V$  is a strictly positive continuous linear functional. The following conditions are necessary and sufficient for the absence of approximate arbitrage:

$$\beta_n > 0 \quad \forall n \in \mathbb{N} \quad \text{and} \quad \bar{\beta} = 1. \quad (4.5)$$

We address sufficiency first. If  $\bar{\beta} = 1$ , then  $V(z) = \int_X z d\lambda$  is a valuation operator on  $(L_1(\lambda), \|\cdot\|_1^{\lambda})$  and hence guarantees the absence of approximate arbitrage opportunities.

We show  $\bar{\beta} = 1$  is necessary by presenting an approximate arbitrage in  $C(\mathbb{N}_{\infty}) \subset L_1(\lambda)$  if (4.3) holds and  $\bar{\beta} < 1$ . Let  $z_n = \sum_{i=1}^n \mathbf{e}_i - \gamma_n \tilde{\mathbf{e}}_n$ , where

$$\gamma_n := \sum_{i=1}^n \frac{\beta_i}{\tilde{\beta}_n + (1 - \bar{\beta})}.$$

Note  $V(z_n) = 0$  and  $z_n \xrightarrow{\lambda} \mathbf{1}_{\mathbb{N}}$ . If  $\|z_n - \mathbf{1}_{\mathbb{N}}\|_1^{\lambda} \rightarrow 0$ , then there is an approximate arbitrage. Let  $\mu_n = T_{\lambda}^{-1}(z_n) = \sum_{i=1}^n \beta_i \delta_i - \gamma_n \sum_{i=n+1}^{\infty} \beta_i \delta_i$ . Then  $\|z_n - \mathbf{1}_{\mathbb{N}}\|_1^{\lambda} \rightarrow 0 \iff \|\mu_n - \lambda\| \rightarrow 0$ . For any  $f \in C(\mathbb{N}_{\infty})$ ,

$$\langle f, \mu_n \rangle = \sum_{i=1}^n f(i) \beta_i - \gamma_n \tilde{\beta}_n \sum_{i=n+1}^{\infty} f(i) \left( \frac{\beta_i}{\tilde{\beta}_n} \right).$$

Consequently,

$$\lim_{n \rightarrow \infty} \langle f, \mu_n \rangle = \langle f, \lambda \rangle - \left( \lim_{n \rightarrow \infty} \gamma_n \tilde{\beta}_n \right) \langle f, \delta_{\infty} \rangle,$$

<sup>23</sup>Note  $a_0 + \sum_{i=1}^{\infty} a_i \beta_i = \sum_{i=1}^{\infty} (a_0 + a_i) \beta_i + a_0 (1 - \bar{\beta})$  where  $a_0 = \text{Lim}[z]$  for  $z \in \mathbb{M}$ .

where

$$\lim_{n \rightarrow \infty} \gamma_n \tilde{\beta}_n = \lim_{n \rightarrow \infty} \frac{\bar{\beta} - \tilde{\beta}_n}{1 + \tilde{\beta}_n^{-1}(1 - \bar{\beta})} = \begin{cases} 0 & \bar{\beta} < 1 \\ 1 & \bar{\beta} = 1. \end{cases}$$

Therefore,  $\bar{\beta} < 1 \implies \|\mu_n - \lambda\| \rightarrow 0$  which constitutes an approximate arbitrage.

*Redefine the bond's payout.* If  $\bar{\beta} < 1$ , we can avoid approximate arbitrage opportunities by redefining the bond's payout. (We are not always free to do this.) As a first step, we can identify the payouts of the bond and the A–D securities with elements of  $B_\lambda$ :  $T_\lambda^{-1}(\mathbf{1}_\mathbb{N}) = \lambda$  and  $T_\lambda^{-1}(e_n) = \beta_n \delta_n$ . Next, let the payout to the bond be redefined as

$$\xi := \lambda + (1 - \bar{\beta}) \delta_\infty. \quad (4.6)$$

Then  $\mu \mapsto \langle \mathbf{1}, \mu \rangle$  is a valuation operator. Note  $\langle \mathbf{1}, \beta_n \delta_n \rangle = \beta_n$  and  $\langle \mathbf{1}, \xi \rangle = \langle \mathbf{1}, \lambda \rangle + (1 - \bar{\beta}) \langle \mathbf{1}, \delta_\infty \rangle = 1$ . Note that we can approximate  $\xi$  by  $\lambda + (1 - \bar{\beta}) \tilde{\delta}_n$ .

It is possible to redefine the payout to the bond in such a way as to achieve the result in the preceding paragraph without placing any weight directly on  $\{\infty\}$ . Let  $C_b(X)$  denote the space of bounded continuous functions on a normal Hausdorff space  $X$  and let  $ba_n(\mathcal{A}_X)$  denote the space of normal charges on the algebra generated by the open sets of  $X$ . Then a version of the Riesz Representation Theorem states that the dual of  $C_b(X)$  is  $ba_n(\mathcal{A}_X)$ .<sup>24</sup> Since  $\mathbb{N}_\infty$  is a normal Hausdorff space and  $C(\mathbb{N}_\infty) = C_b(\mathbb{N}_\infty)$ , the theorem applies here. The pure charge that represents the linear functional  $f \mapsto (1 - \bar{\beta}) \text{Lim}[f]$ , for  $f \in C(\mathbb{N}_\infty)$ , does not involve placing any weight on  $\{\infty\}$ . Nevertheless, the effect is the same.

## 5. DYNAMIC SECURITIES MARKET MODEL IN THE COUNTABLE SETTING

We build on the setting in Section 4, adding a filtration. This is essentially the stochastic setting in the example of Back and Pliska (1991).

We formalize the setting for the doubling strategy as outlined by Harrison and Kreps (1979). We consider a roulette wheel with two colors, RED and BLACK, that will be spun repeatedly. There is positive probability that each color will occur on the next spin.

**Stochastic processes.** Given the measurable space  $(\mathbb{N}_\infty, \mathcal{B}(\mathbb{N}_\infty))$ , a random variable  $Z$  is a measurable function  $Z : \mathbb{N}_\infty \rightarrow \mathbb{R}$ . Such a random variable can be characterized by the sequence  $\{Z(\omega)\}_{\omega \in \mathbb{N}_\infty}$ , where  $Z(\omega)$  is the value of  $Z$  given that RED occurs on the  $\omega$ -th spin for  $\omega \in \mathbb{N}$  and  $Z(\infty)$  is the value of  $Z$  if RED never occurs. Given  $S$ , a collection of subsets of  $\mathbb{N}_\infty$ , let  $\sigma(S)$  denote the  $\sigma$ -algebra generated by  $S$ . Define

$$\mathcal{F}_i := \sigma(\{\{1\}, \{2\}, \dots, \{i\}\}) \quad i \in \mathbb{N}. \quad (5.1)$$

Note  $\{\mathcal{F}_i\}_{i=1}^\infty$  is a filtration—an increasing family of sub- $\sigma$ -algebras such that  $\mathcal{F}_n \subset \mathcal{F}_{n+1}$  for all  $n \in \mathbb{N}$ . Define  $\mathcal{F}_0 := \{\emptyset, \mathbb{N}_\infty\}$  and  $\mathcal{F}_\infty := \sigma(\cup_{i=1}^\infty \mathcal{F}_i)$ . Note  $\mathcal{F}_\infty = \mathcal{B}(\mathbb{N}_\infty)$ .

<sup>24</sup>Theorem 13.10 in Aliprantis and Border (1999).

A stochastic process  $\mathcal{Z} = \{Z_i\}_{i=1}^\infty$  is an infinite sequence of random variables. For fixed  $i$ ,  $Z_i$  describes the value of the process  $\mathcal{Z}$  after the  $i$ -th spin. We restrict our attention to stochastic processes that are adapted to the filtration—that is, such that for each  $i \geq 1$ ,  $Z_i$  is measurable with respect to  $\mathcal{F}_i$ . Note that  $Z_i \in C(\mathbb{N}_\infty)$ , since  $Z_i$  has a constant tail. Every adapted stochastic process  $\mathcal{Z}$  can be described in terms of two sequences:  $\{z_r(\omega)\}_{\omega \in \mathbb{N}}$  and  $\{z_b(i)\}_{i \in \mathbb{N}}$ , where  $z_r(\omega)$  is the terminal value of  $\mathcal{Z}$  given the first RED occurs on spin  $\omega$  and  $z_b(i)$  is the value of  $\mathcal{Z}$  after the  $i$ -th spin given that RED has not yet occurred:

$$Z_i(\omega) = \begin{cases} z_r(\omega) & \omega \leq i \\ z_b(i) & \omega > i. \end{cases} \quad (5.2)$$

Note that if  $z_b(i) = z_r(i)$ , then  $Z_i$  is  $\mathcal{F}_{i-1}$ -measurable. If  $z_b(i) = z_r(i) = \zeta_i$  for all  $i \in \mathbb{N}$ , then we say that  $\mathcal{Z}$  is *predictable* (or *previsible*) and we say the sequence  $\{\zeta_i\}_{i \in \mathbb{N}}$  represents a predictable stochastic process.

For fixed  $i$ ,  $Z_i(\cdot)$  is a random variable; for fixed  $\omega$ ,  $Z_i(\omega)$  is a *path*. On each path,  $Z_i(\omega)$  converges to  $z_r(\omega)$ . We can also express  $Z_i$  in terms of unit coordinate vectors:

$$Z_i = \sum_{j=1}^i z_r(j) \mathbf{e}_j + z_b(i) \tilde{\mathbf{e}}_i. \quad (5.3)$$

Given (5.3), we have

$$\Delta Z_i = (z_r(i) - z_b(i-1)) \mathbf{e}_i + (z_b(i) - z_b(i-1)) \tilde{\mathbf{e}}_i, \quad (5.4)$$

where  $\Delta Z_i := Z_i - Z_{i-1}$ .

Fix the probability measure  $\varphi := \sum_{i=1}^\infty \alpha_i \delta_i$ , where  $\alpha_i > 0$  for all  $n \in \mathbb{N}$  and  $\sum_{i=1}^\infty \alpha_i = 1$ . Let  $\tilde{\alpha}_n := \sum_{i=n+1}^\infty \alpha_i$ . Note  $\varphi$  is equivalent to  $\lambda$  ( $\varphi \ll \lambda$  and  $\lambda \ll \varphi$ ). In particular,  $B_\varphi = B_\lambda$ , the  $\varphi$ -null sets are  $\emptyset$  and  $\{\infty\}$ , and  $\text{supp } \varphi = \mathbb{N}_\infty$ . Note  $(\mathbb{N}_\infty, \mathcal{B}(\mathbb{N}_\infty), \varphi)$  is a probability space and  $(\mathbb{N}_\infty, \mathcal{B}(\mathbb{N}_\infty), \{\mathcal{F}_i\}_{i=1}^\infty, \varphi)$  is a filtered probability space. Define  $L_1(\varphi) := L_1(\mathbb{N}_\infty, \mathcal{B}(\mathbb{N}_\infty), \varphi)$ . For  $\mu \in B_\varphi$ ,  $T_\varphi(\mu) = d\mu/d\varphi = \sum_{i=1}^\infty (m_i/\alpha_i) \mathbf{e}_i$ . For  $z \in L_1(\varphi)$ ,  $T_\varphi^{-1}(z) = \sum_{i=1}^\infty \alpha_i z(i) \delta_i$ . Let  $\pi = T_\varphi(\lambda) = d\lambda/d\varphi = \sum_{i=1}^\infty (\beta_i/\alpha_i) \mathbf{e}_i$ . Note  $E^\varphi[\pi] = \sum_{i=1}^\infty \pi(i) \alpha_i = \bar{\beta}$ . Let  $E^\varphi[Z] = \sum_{i=1}^\infty Z(i) \alpha_i$  for  $Z \in L_1(\varphi)$ . Let  $E_{i-1}^\varphi[Z] := E^\varphi[Z \mid \mathcal{F}_{i-1}]$  for  $\mathcal{F}_{i-1}$ -measurable  $Z \in L_1(\varphi)$ . Finally, let  $p_i$  denote the conditional probability of  $\omega = i$  given  $\omega \geq i$ :

$$p_i = \frac{\alpha_i}{1 - \sum_{j=1}^{i-1} \alpha_j} = \frac{\alpha_i}{\tilde{\alpha}_i} \quad \text{and} \quad \alpha_i = p_i \prod_{j=1}^{i-1} (1 - p_j).$$

( $\alpha_i$  is the unconditional probability of  $\omega = i$ .)

Now we construct a *shock process*  $\mathcal{U} := \{U_i\}_{i=0}^\infty$  to use as a building block. Define  $\zeta_i := \sqrt{(1-p_i)/p_i}$ . Let  $U_i := \sum_{j=1}^i u_r(j) \mathbf{e}_j + u_b(i) \tilde{\mathbf{e}}_i$ , where

$$\begin{aligned} u_r(i) &= u_b(i-1) + \zeta_i \\ u_b(i) &= u_b(i-1) - \zeta_i^{-1} \end{aligned}$$

subject to  $u_b(0) = 0$ . Note  $\Delta U_i = \zeta_i e_i - \zeta_i^{-1} \tilde{e}_i$ .  $\mathcal{U}$  is a  $\varphi$ -martingale:  $E_{i-1}^\varphi[\Delta U_i] = 0$ . In addition,

$$E_{i-1}^\varphi[(\Delta U_i)^2](\omega) = \begin{cases} 0 & \omega < i \\ 1 & \omega \geq i. \end{cases}$$

**Dynamic securities market.** Consider a market for trading two securities at a countable number of times  $0 = t_0 < t_1 < \dots < T$ .<sup>25</sup> The price of one security (the bond or ‘money-market account’) at time  $t_i$  equals 1 in every state:  $B_i(\omega) = 1$ . The state-by-state price of the second security (the ‘stock’) at time  $t_i$  is denoted  $S_i(\omega)$ .<sup>26</sup>

The dynamics of the stock price can be specified by

$$\Delta S_i = m_i + \sigma_i \Delta U_i, \quad (5.5)$$

given some  $S_0$ . We assume  $\sigma_i \neq 0$  for all  $i \in \mathbb{N}$  (spanning condition). The mean and the variance of the conditional change in the stock price (conditional on  $\omega \geq i$ ) are given by

$$E_{i-1}^\varphi[\Delta S_i](\omega \geq i) = m_i \quad \text{and} \quad E_{i-1}^\varphi[(\Delta S_i - m_i)^2](\omega \geq i) = \sigma_i^2.$$

Given (5.5), we have

$$S_i = S_0 + \sum_{j=1}^i \Delta S_j = S_0 + \sum_{j=1}^i m_j + \sum_{j=1}^i \sigma_j \Delta U_j$$

so that

$$\begin{aligned} s_r(i) &= s_b(0) + \sum_{j=1}^i m_j - \sum_{j=1}^{i-1} \sigma_j \zeta_j^{-1} + \sigma_i \zeta_i \\ s_b(i) &= s_b(0) + \sum_{j=1}^i m_j - \sum_{j=1}^i \sigma_j \zeta_j^{-1}. \end{aligned}$$

**Price of risk, state-price deflator, and change-of-measure process.** Let  $\ell_i := m_i/\sigma_i$ , which is the coefficient of variation (the ratio of the mean to the standard deviation) conditional on  $\omega \geq i$ . Since the interest rate is zero,  $\ell_i$  is the *Sharpe ratio* of the stock. It is also the *price of risk*. (Note  $\{m_i\}_{i=1}^\infty$ ,  $\{\sigma_i\}_{i=1}^\infty$ , and  $\{\ell_i\}_{i=1}^\infty$  represent predictable processes.)

Define  $\mathcal{Y} := \{Y_i\}_{i=1}^\infty$ . Let  $Y_i := \prod_{j=1}^i X_j$  where  $X_i := 1 - \ell_i \Delta U_i$ . Note  $E_{i-1}^\varphi[X_i] = 1$  and  $E_{i-1}^\varphi[X_i \Delta U_i](\omega \geq i) = -\ell_i$  since  $X_i \Delta U_i = \Delta U_i - \ell_i (\Delta U_i)^2$ . Note that  $\mathcal{Y}$  is a  $\varphi$ -martingale:  $E_{i-1}^\varphi[\Delta Y_i] = 0$  since  $\Delta Y_i = -Y_{i-1} \ell_i \Delta U_i$ . In particular,  $E^\varphi[Y_n] = 1$  for all  $n \in \mathbb{N}$ .

Note that  $\{S_i Y_i\}_{i=1}^\infty$  is also a  $\varphi$ -martingale:

$$E_{i-1}^\varphi[\Delta(S_i Y_i)] = Y_{i-1} E_{i-1}^\varphi[X_i \Delta S_i] + S_{i-1} E_{i-1}^\varphi[\Delta Y_i] = 0,$$

<sup>25</sup>For example,  $T < \infty$  and  $t_i = T(1 - (1/2)^i)$ .

<sup>26</sup>To allow for a non-zero interest rate  $r$ , let  $\tilde{B}_i(\omega) = e^{rt_i}$  be the money-market account and define  $\tilde{S}_i := \tilde{B}_i S_i$ . Then  $B_i = \tilde{B}_i/\tilde{B}_i$  and  $S_i = \tilde{S}_i/\tilde{B}_i$ .

since (conditional on  $\omega \geq i$ )

$$E_{i-1}^\varphi[X_i \Delta S_i] = m_i + \sigma_i E_{i-1}^\varphi[X_i \Delta U_i] = m_i - \ell_i \sigma_i = 0.$$

It is convenient to reexpress  $X_i$  and  $Y_n$ . In particular,

$$X_i = \sum_{\omega=1}^{i-1} e_\omega + \left(\frac{q_i}{p_i}\right) e_i + \left(\frac{1-q_i}{1-p_i}\right) \tilde{e}_i, \quad (5.6)$$

where

$$q_i := (1 - \ell_i \zeta_i) p_i. \quad (5.7)$$

Thus  $\ell_i = (p_i - q_i)/\sqrt{p_i(1-p_i)}$ . Note that  $q_i$  can be computed directly from  $s_r$  and  $s_b$ :

$$q_i = \frac{s_b(i) - s_b(i-1)}{s_b(i) - s_r(i)}.$$

The condition  $\sigma_i \neq 0$  is equivalent to  $s_b(i) - s_r(i) \neq 0$ .<sup>27</sup> Given (5.6), we can write

$$Y_n = \sum_{i=1}^n y_r(i) e_i + y_b(n) \tilde{e}_n,$$

where

$$y_r(i) = \frac{q_i \prod_{j=1}^{i-1} (1 - q_j)}{p_i \prod_{j=1}^{i-1} (1 - p_j)} = \frac{\beta_i}{\alpha_i}$$

$$y_b(n) = \frac{\prod_{i=1}^n (1 - q_i)}{\prod_{i=1}^n (1 - p_i)} = \frac{\tilde{\beta}_n + (1 - \bar{\beta})}{\tilde{\alpha}_n},$$

using  $\beta_i = q_i \prod_{j=1}^{i-1} (1 - q_j)$ .

If  $\mathcal{Y}$  is strictly positive martingale, then it is a state-price deflator.<sup>28</sup> It is easily seen that  $\mathcal{Y}$  is strictly positive if and only if

$$0 < q_i < 1 \quad \forall i \in \mathbb{N}. \quad (5.8)$$

Note (5.8) is equivalent to (4.3). If Condition (5.8) holds, we can compute the equivalent measure  $\lambda = T_\varphi^{-1}(y_r)$ , where  $y_r = \pi$ . (Note  $Y_n \xrightarrow{\varphi\text{-a.e.}} \pi$ .)

A uniformly integrable state-price deflator is a change-of-measure process. Note

$$T_\varphi^{-1}(Y_n) = \sum_{i=1}^n \beta_i \delta_i + (\tilde{\beta}_n + (1 - \bar{\beta})) \sum_{i=n+1}^{\infty} \left(\frac{\alpha_i}{\tilde{\alpha}_n}\right) \delta_i,$$

<sup>27</sup>Given  $s_r$  and  $s_b$ , we can write

$$m_i = s_r(i) - s_b(i-1) p_i + s_b(i) - s_b(i-1) (1-p_i)$$

$$= (p_i - q_i) s_b(i) - s_r(i)$$

$$\sigma_i = s_r(i) - s_b(i-1) - m_i^2 p_i + s_b(i) - s_b(i-1) - m_i^2 (1-p_i)^{1/2}$$

$$= p_i (1-p_i) s_b(i) - s_r(i)^2^{1/2}.$$

<sup>28</sup>Warning: Harrison and Kreps (1979) and others include uniform integrability in their definition of a state-price deflator (which we call a change-of-measure process).

and therefore

$$T_\varphi^{-1}(Y_n) \xrightarrow{w^*} \xi, \quad (5.9)$$

where  $\xi = \lambda + (1 - \bar{\beta}) \delta_\infty$  [see (4.6)]. Therefore,  $\mathcal{Y}$  is UI if and only if  $\bar{\beta} = 1$ . Given  $\bar{\beta} = 1$ ,  $E^\lambda[Z] = E^\varphi[y_r Z]$  and  $E_{i-1}^\lambda[Z_i] = E_{i-1}^\lambda[X_i Z_i]$ . In this case, we say  $\lambda$  is an equivalent martingale measure.

**Self-financing trading strategies.** A trading strategy is a pair of predictable processes represented by  $(\{\theta_i\}_{i=1}^\infty, \{\phi_i\}_{i=1}^\infty)$  where  $\theta_i$  is the number of bonds and  $\phi_i$  is the number of shares of stock held at time  $t_i$  after any changes in the value of the stock but before any rebalancing. The exposition that follows is intended to make the meaning of this clear.

Let  $G_0$  denote the initial amount invested. At time  $t_0$ ,  $G_0$  is apportioned between the stock and the bond:  $G_0 = \theta_1 + \phi_1 S_0$ . At time  $t_1$ , after any changes in the share price but before any rebalancing, the value of the portfolio is  $G_1 = \theta_1 + \phi_1 S_1$ . After rebalancing, the value of the portfolio is  $G'_1 = \theta_2 + \phi_2 S_1$ . In general,  $G_i = \theta_i + \phi_i S_i$  and  $G'_i = \theta_{i+1} + \phi_{i+1} S_i$ . For a self-financing trading strategy,  $G'_i = G_i$  which implies

$$(\theta_{i+1} - \theta_i) + (\phi_{i+1} - \phi_i) S_i = 0 \quad \forall i \in \mathbb{N}. \quad (5.10)$$

In other words, any change in the value of the stock holdings that comes from rebalancing is offset by an equal change in the opposite direction of the value of the bond holdings. From one period to the next, the change in the value of a portfolio generated by a self-financing trading strategy is  $\Delta G_i = \phi_i \Delta S_i$  and therefore  $G_i = G_0 + \sum_{j=1}^i \phi_j \Delta S_j$ . We refer to  $G_i$  as the *gain* and  $\mathcal{G} = \{G_i\}_{i=1}^\infty$  as the gains process.<sup>29</sup> A self-financing trading strategy is characterized by the pair  $(G_0, \{\phi_i\}_{i=1}^\infty)$ .

A *finite trading strategy* is a self-financing trading strategy for which  $\phi_i = 0$  for all  $i > n \in \mathbb{N}$ . The gain  $G_n$  generated by a finite trading strategy is called a *finite gain* stopped at  $n$ . Let  $G_i = \sum_{j=1}^i g_r(j) e_j + g_b(i) \tilde{e}_i$ , where

$$g_r(i) = g_b(i-1) + \phi_i (s_r(i) - s_b(i-1)) \quad (5.11a)$$

$$g_b(i) = g_b(i-1) + \phi_i (s_b(i) - s_b(i-1)), \quad (5.11b)$$

subject to  $g_b(0) = G_0$ . For a finite trading strategy stopped at  $n$ ,  $G_i = g_r$  for all  $i \geq n$ . Consequently,  $G_i \rightarrow g_r$  in all modes of convergence.

We now describe the space of marketed payouts and show that it is the space of finite gains. Define  $\mathbb{M}_{0,n} := B \cup (\bigcup_{0 < i \leq n} S_i)$  and the set of marketed securities as  $\mathbb{M}_0 := \bigcup_{n \geq 0} \mathbb{M}_{0,n}$ . The given prices for the marketed securities are  $V_0(B) = 1$  and  $V_0(S_i) = S_0$  for all  $i \in \mathbb{N}$ . The space of marketed payouts,  $\mathbb{M} := \mathbf{sp}(\mathbb{M}_0)$ . Now  $z \in \mathbf{sp}(\mathbb{M}_{0,n})$  has the form  $z = a_0 B + \sum_{i=1}^n a_i S_i$ , where  $V(z) = a_0 V_0(B) + \sum_{i=1}^n a_i V_0(S_i) = a_0 + S_0 \sum_{i=1}^n a_i$ . This  $z$  can be obtained via the following finite self-financing trading strategy:  $G_0 = V(z)$  and  $\phi_i = \sum_{j=i}^n a_j$ .

It turns out that set of marketed securities in the dynamic case spans the same space as the set of marketed securities in the static case above, so that the two

<sup>29</sup>Warning: Pliska (1997) and others refer to  $G_i - G_0$  as the gain.

spaces of marketed payouts are identical. Therefore, it is not surprising that the conditions for no arbitrage and no approximate arbitrage are also identical.

*Arbitrage.* An arbitrage is a finite trading strategy for which  $G_0 = 0$ ,  $g_r(i) \geq 0$  for all  $i \in \mathbb{N}$ , and  $g_r(i) > 0$  for some  $i \in \mathbb{N}$ .

Consider the finite trading strategy where  $G_0 = 0$  and

$$\phi_i = \begin{cases} \frac{1}{s_r(n) - s_b(n)} & i = n \\ 0 & i \neq n. \end{cases}$$

This trading strategy produces  $g_r = (1 - q_n) e_n - q_n \tilde{e}_n$ , which is an arbitrage unless Condition (5.8) holds.

Conversely, suppose Condition (5.8) holds. Set  $\beta_i = q_i \prod_{j=1}^{i-1} (1 - q_j)$ .<sup>30</sup> Then for any finite gain stopped at  $n$ , we can use (5.11) to eliminate  $\{\phi_i\}_{i=1}^n$  and  $\{g_b(i)\}_{i=1}^{n-1}$  to produce:

$$\sum_{i=1}^n g_r(i) \beta_i + g_b(n) (\tilde{\beta}_n + (1 - \bar{\beta})) = G_0, \quad (5.12)$$

where  $g_r(j) = g_b(n)$  for  $j \geq n + 1$ . Given the positivity of  $\beta_i$  and  $\tilde{\beta}_n + (1 - \bar{\beta})$ , no finite gain can be an arbitrage. Note that we can express (5.12) as  $V(g_r) = G_0$ , where  $V$  is given in (4.4).

*Approximate arbitrage.* Assume the no-arbitrage Condition (5.8) holds. Therefore, a state-price deflator exists and  $\lambda = \sum_{i=1}^{\infty} \beta_i \delta_i$  is an equivalent measure. (See above.) Given  $\bar{\beta} = 1$ ,  $\lambda$  is an equivalent martingale measure and the gains process is a martingale:  $E_{i-1}^\lambda[\Delta G_i] = \phi_i E_{i-1}^\lambda[\Delta S_i] = \phi_i E_{i-1}^\varphi[X_i \Delta S_i] = 0$ . Therefore,  $E^\lambda[G_n] = G_0$  where  $E^\lambda[G_n] = \int_X G_n d\lambda$ .

An *admissible* self-financing trading strategy is one for which the generated gain process converges in the appropriate topology. For example, if the payout space is  $(L_1(\lambda), \|\cdot\|_1^\lambda)$ , then an admissible trading strategy is one for which  $\|G_n - G\|_1^\lambda \rightarrow 0$  for some  $G \in L_1(\lambda)$ . In this case,  $z \mapsto E^\lambda[z]$  is a valuation operator. On the other hand, if the payout space is  $(M(X), w^*)$ , then an admissible trading strategy is one for which  $T_\lambda^{-1}(G_n) \xrightarrow{w^*} \mu \in M(X)$ , where  $T_\lambda(\mu) = d\mu/d\lambda$ . In this case  $\mu \mapsto \langle \mathbf{1}, \mu \rangle$  is a valuation operator. For  $\mu \in B_\lambda$ ,  $\langle \mathbf{1}, \mu \rangle = E^\lambda[T_\lambda^{-1}(\mu)]$ .

*Equivalences.* The following statements are equivalent:

- (1)  $\mathcal{Y}$  is a strictly positive and uniformly integrable  $\varphi$ -martingale.
- (2) There is an equivalent martingale measure  $\lambda$ .
- (3)  $z \mapsto \int_X z d\lambda$  is a valuation operator for  $(L_1(\lambda), \|\cdot\|_1^\lambda)$ .

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<sup>30</sup>Note  $q_n$  is the *conditional price* of bet that pays \$1 if RED occurs (conditional on  $\omega \geq n$ ), while  $\beta_n$  is the *unconditional price* (i.e., the price given before the first spin; conditional on  $\omega \geq 1$ ).

## 6. ROLE REVERSAL IN THE COUNTABLE SETTING

In this section, we reverse the roles of the dual pair of spaces: Let  $C(\mathbb{N}_\infty)$  equipped with the sup norm topology be the space of payouts and let  $M(\mathbb{N}_\infty)$  be the space of price systems. The valuation operator is given by

$$f \mapsto \langle f, \mu \rangle = \sum_{i=1}^{\infty} f(i) m_i + \text{Lim}[f] m_\infty,$$

for some strictly positive  $\mu$  where  $m_i > \infty$  for  $i \in \mathbb{N}_\infty$ .

**Static setting.** Let us revisit the conditions for no arbitrage and no approximate arbitrage in the static setting of Section 4. In this case, Conditions (4.3) are necessary and sufficient for the existence of a valuation operator.

Take  $\xi = \lambda + (1 - \bar{\beta}) \delta_\infty$  [as given in (4.6)] to represent the *price system*. Then, given Conditions (4.3),  $V : C(\mathbb{N}_\infty) \rightarrow \mathbb{R}$  is a valuation operator, where

$$V(f) = \langle f, \xi \rangle = \langle f, \lambda \rangle + (1 - \bar{\beta}) \langle f, \delta_\infty \rangle = \int_X f d\lambda + \text{Lim}[f] (1 - \bar{\beta}).$$

Compare with (4.4). The payout to the  $n$ -th Arrow–Debreu security is  $e_n$  and the payout to the bond be given by  $\mathbf{1}$ . We have  $\langle e_n, \xi \rangle = \beta_n$  and  $\langle \mathbf{1}, \xi \rangle = 1$ .

**Dynamic setting.** In this setting, a trading strategy is admissible if and only if it generates a gain process that converges in the sup norm topology.<sup>31</sup>

Note  $G_n \in C(\mathbb{N}_\infty)$ , assuming  $G_n(\infty) = \lim_{i \rightarrow \infty} G_n(i)$ . If  $\{G_n\}_{n=1}^\infty$  converges in the sup norm topology, then  $G_n \xrightarrow{\|\cdot\|_\infty} g_r \in C(\mathbb{N}_\infty)$ , in which case  $\lim_{n \rightarrow \infty} g_r(n) - g_b(n) = 0$ . Given  $G_0$  and  $g_r$ , we can eliminate  $\phi_i$  from (5.11) and solve recursively for

$$g_b(i) = \frac{G_0 - \sum_{j=1}^i g_r(j) \beta_j}{\tilde{\beta}_i + (1 - \bar{\beta})}. \quad (6.1)$$

Assuming (5.8) and referring to (6.1), note

$$\lim_{n \rightarrow \infty} g_r(n) - g_b(n) = \text{Lim}[g_r] - \frac{G_0 - \sum_{i=1}^\infty g_r(i) \beta_i}{1 - \bar{\beta}}.$$

Therefore

$$G_n \xrightarrow{\|\cdot\|_\infty} g_r \iff \langle g_r, \xi \rangle = G_0.$$

Consequently,  $g_r = \mathbf{1}$  combined with  $G_0 = 0$  does not produce an approximate arbitrage because the implied gains process does not converge. Thus  $f \mapsto \langle f, \xi \rangle$  is a valuation operator.

In this case,  $\mathcal{Y}$  is a state-price deflator, although it is not uniformly integrable as can be seen in (5.9) and it does not deliver a change-of-measure process (as we have defined it). Thus,  $\xi$  is a martingale measure, but it is not equivalent to

<sup>31</sup>Back and Pliska (1991) present the following stock-price dynamics:  $s_r(i) = 2^{-i} (i^2 + 2i + 2)$  and  $s_b(i) = 2^{-i}$ . These dynamics produce  $q_i = 1/(i+1)^2$ ,  $\beta_i = (2i(i+1))^{-1}$ , and  $\bar{\beta} = 1/2$ . Other stock-price dynamics produce identical bet prices; for example,  $s_r(i) = 2$  and  $s_b(i) = 2/(i+2)$ . Even though the bet prices are identical, the trading strategies  $\{\phi_i\}_{i=1}^\infty$  required to produce a given payout are quite different.

$\varphi$ . Nevertheless, (i)  $\int_X G_n d\xi = E^\varphi[G_n Y_n]$  and (ii) if  $\|G_n - g_r\|_\infty \rightarrow 0$ , then  $\int_X g_r d\xi = \lim_{n \rightarrow \infty} E^\varphi[G_n Y_n]$ .

*Equivalences.* The following statements are equivalent:

- (1)  $\mathcal{Y}$  is a strictly positive  $\varphi$ -martingale.
- (2) There is a valuation operator for  $(C(X), \|\cdot\|_\infty)$ .

## 7. AN INTRODUCTORY APPLICATION TO BLACK–SCHOLES

Assume the Black–Scholes model holds. Let  $C(K, S_t, T - t)$  be the value at time  $t$  of a European call option with a strike price  $K$  that matures at time  $T$  given the current stock price of  $S_t$ . In particular,

$$C(K, s, \tau) = s \Phi(z) - e^{-r\tau} K \Phi(z - \sigma \sqrt{\tau}),$$

with

$$z = \frac{\log(s/K) + (r + \sigma^2/2)\tau}{\sigma \sqrt{\tau}},$$

where  $\Phi(x) = \frac{1}{2} (1 + \operatorname{erf}(x/\sqrt{2}))$  is the cumulative standard normal distribution function.<sup>32</sup> The call option price  $C$  satisfies the partial differential equation (PDE)

$$r C(K, s, \tau) = -\frac{\partial}{\partial \tau} C(K, s, \tau) + r s \frac{\partial}{\partial s} C(K, s, \tau) + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2}{\partial s^2} C(K, s, \tau), \quad (7.1)$$

subject to the boundary condition  $C(K, s, 0) = (s - K)^+$ , where  $x^+ := x \vee 0$ .

Define

$$G(K, s, \tau) := \frac{\partial^2}{\partial K^2} C(K, s, \tau).$$

Letting  $C = G$  in (7.1), we see that  $G$  satisfies the PDE. In fact,  $G$  is called the *fundamental solution* to the PDE; it is also known as the *Green's function*.<sup>33</sup> As Breeden and Litzenberger (1978) point out,  $G(K, S_t, T - t)$  is the value at time  $t$  (given the stock price of  $S_t$ ) of an Arrow–Debreu security that pays at time  $T$  one unit in the event  $S_T = K$ . In fact, for  $t \in (0, T]$ ,<sup>34</sup> one may verify

$$C(K, S_0, T) = \int_0^\infty C(K, x, T - t) G(x, S_0, t) dx. \quad (7.2)$$

<sup>32</sup>See Duffie (2001, Chapter 5) for the derivation of the Black–Scholes call price.

<sup>33</sup>The boundary condition can be obtained by twice differentiating (in a generalized sense) the boundary condition for the call option with respect to the strike price. The resulting ‘density’ is called the Dirac delta function. (Warning: the Dirac delta function is typically denoted  $\delta_K(x)$ . Do not confuse this with the point mass measure  $\delta_x$ . Of course they are related: The Dirac delta is the ‘density’ of the point mass with respect to Lebesgue measure.) The Fourier transform of the delta function can be obtained as follows. Given the uniform distribution with support  $[K - \alpha, K + \alpha]$ , compute its characteristic function:  $e^{iKt} \sin(\alpha t)/(\alpha t)$ , where  $i$  is the imaginary unit. The limit as  $\alpha \rightarrow 0$  is  $e^{iKt}$ . The inverse Fourier transform of  $e^{iKt}$  is the delta function.

<sup>34</sup>For  $t = 0$ , the expression is formally correct when  $G(x, S_0, 0)$  is interpreted as a Dirac delta function.

We now place the preceding in our setting. Let  $X = [0, \infty]$ . Let the reference measure be Lebesgue measure and the pricing function be  $G(\cdot, S_0, T)$ . In other words

$$\lambda(B) = \int_B G(x, S_0, T) \text{Leb}(dx) \quad \forall B \in \mathcal{B}([0, \infty]).$$

We can express the value of any payout  $\mu \ll \lambda$  in terms of its density  $z = d\mu/d\lambda$ :  $\mu(X) = \int_X z d\lambda$ . For example, the density of the call option is  $z(x) = (x - K)^+$ . The density of a zero-coupon bond is  $z(x) = 1$  and the value of the bond is  $\int_X \mathbf{1} d\lambda = e^{-rT}$ . Define the equivalent measure  $\varphi$  via  $d\varphi = e^{rT} d\lambda$ . Then (continuing to assume  $\mu \ll \lambda$ ),

$$\mu(X) = \int_X e^{rT} z d\varphi = E^\varphi[e^{rT} z].$$

Our goal now is to make closer contact with the martingale formulation of asset prices. Consider an increasing sequence  $\{t_n\}_{n=1}^\infty$  such that  $t_1 > 0$  and  $t_n \rightarrow T$ . Define  $\lambda_n$  via  $\lambda_n(dx) = G(x, S_0, t_n) \text{Leb}(dx)$  and define  $w_n(x) = C(K, x, T - t_n)$ . Then we can reexpress (7.2) evaluated at  $t = t_n$  as

$$C(K, S_0, T) = \int_X w_n(x) d\lambda_n(dx)$$

Note  $\lambda_n(X) = e^{-rt_n}$ . Define  $\varphi_n$  via  $d\varphi_n = e^{rt_n} d\lambda_n$ . Let  $\beta_n := e^{rt_n}$  be the value of the money market account at  $t_n$ . Then

$$C(K, S_0, T) = E^{\varphi_n}[w_n/\beta_n].$$

We are led to indexing the probability measure  $\varphi_n$  as a substitute for an explicit filtration, which is absent in this purely static analysis.<sup>35</sup>

Let us turn for the moment to an explicitly dynamic aspect of the Black–Scholes model. There is a self-financing trading strategy that replicates the European put option. Let the value of the money-market account at time  $t$  be given by  $\beta_t = e^{rt}$ . The self-financing trading strategy that replicates a call option is given by

$$C(K, S_t, T - t) = a_t S_t + b_t \beta_t,$$

where

$$a_t = \frac{\partial}{\partial s} C(K, s, T - t) \Big|_{s=S_t}$$

$$b_t = \frac{C(K, S_t, T - t) - a_t S_t}{\beta_t}.$$

This trading strategy combined with a sequence of stopping times provides a mechanism for constructing the sequence  $\{w_n\}_{n=1}^\infty$  in such a way as to guarantee that it is adapted to a suitable filtration.

The payout of an Arrow–Debreu security in this setting is not absolutely continuous with respect to either  $\lambda$  or Lebesgue measure. Nevertheless, we can represent the

<sup>35</sup>Let  $B_t$  for  $t \geq 0$  be a standard Brownian motion on a suitable probability space  $(\Omega, \mathcal{F}, P)$ . Given  $S_t = S_0 \exp\{(r - \sigma^2/2)t + \sigma B_t\}$  and  $\beta_t = e^{rt}$ , we have

$$E[C(K, S_{t_n}, T - t_n)/\beta_{t_n}] = C(K, S_0, T),$$

where the expectation is taken with respect to the probability measure  $P$ .

payout as the limit of a sequence of payouts, all of which are absolutely continuous. Define

$$z_n(x) = G(K, x, T - t_n).$$

Note  $z_n(x)$  becomes concentrated on  $K$  as  $n \rightarrow \infty$ . Define  $\mu_n$  via

$$\mu_n(dx) = z_n(x) \lambda_n(dx) = G(K, x, T - t_n) G(x, S_0, t_n) \text{Leb}(dx).$$

Note

$$\mu_n(X) = \int_0^\infty z_n(x) \lambda_n(dx) = E^{\varphi_n} [z_n / \beta_n] = G(K, S_0, T).$$

Finally note  $\mu_n \xrightarrow{w^*} G(K, S_0, T) \delta_K$ , since

$$\begin{aligned} \langle f, \mu_n \rangle &= \int_0^\infty f(x) z_n(x) \lambda_n(dx) \xrightarrow{n} \\ &f(K) G(K, S_0, T) = \langle f, G(K, S_0, T) \delta_K \rangle \quad \forall f \in C([0, \infty]). \end{aligned}$$

There is a self-financing trading strategy that replicates an Arrow–Debreu security is given by

$$G(K, S_t, T - t) = \hat{a}_t S_t + \hat{b}_t \beta_t,$$

where<sup>36</sup>

$$\begin{aligned} \hat{a}_t &= \frac{\partial}{\partial s} G(K, s, T - t) \Big|_{s=S_t} \\ \hat{b}_t &= \frac{G(K, S_t, T - t) - a_t S_t}{\beta_t}. \end{aligned}$$

The question of the integrability of the trading strategy rears its head here. [More to be said here.]

**Continuous-time doubling strategy.** Now consider the continuous-time doubling strategy as described in Karatzas (1997, pp. 3–4).<sup>37</sup>

Fix a standard Brownian motion  $B$  in  $\mathbb{R}$ , restricted to some time interval  $[0, T]$ , on a given probability space  $(X, \mathcal{B}(X), P)$ . Also fix the standard filtration  $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}$  of  $B$ . Let  $S_t$  and  $\beta_t$  denote the values of the stock and the money-market account (respectively) at time  $t$ . Let  $dS_t = S_t dB_t$  with  $S_0 = 1$  and  $\beta_t = 1$  for all  $t$ . (In other words, both the interest rate and the price of risk are zero.) Note both  $S$  and  $\beta$  are martingales. Therefore, there is a state-price deflator  $\pi$ , where  $\pi_t = 1$  for all  $t$ . We believe it can be shown there are no arbitrage opportunities in  $L_1(P)$  equipped with its norm topology. Here, we limit ourselves to demonstrating that the doubling strategy is not an arbitrage.

Define the martingale  $z_t := \int_0^t (T - u)^{-1/2} dB_u$  for  $t \in [0, T]$ . Note  $E[z_t] = 0$  for  $t \in [0, T]$ . Fix  $\alpha > 0$ . Consider the stopping time  $\varsigma = \inf \{t : z_t = \alpha\} \wedge T$ . Note

<sup>36</sup>Note  $\frac{\partial}{\partial s} G(K, s, \tau) = \frac{\partial}{\partial s} \frac{\partial^2}{\partial K^2} C(K, s, \tau) = \frac{\partial^2}{\partial K^2} \frac{\partial}{\partial s} C(K, s, \tau)$ .

<sup>37</sup>Duffie (2001, pp. 103–104) provides a less technical presentation.

$\varsigma \in (0, T)$  almost surely. Consider the self-financing trading strategy  $(a_t, b_t)$ , where

$$\begin{aligned} a_t &= \frac{1}{S_t \sqrt{T-t}} \mathbf{1}_{\{t \leq \varsigma\}} \\ b_t &= \int_0^t a_u dS_u - a_t S_t \end{aligned}$$

Then the value of the portfolio is

$$a_t S_t + b_t \beta_t = \int_0^t a_u dS_u = z_t^\varsigma,$$

where

$$z_t^\varsigma := z_t \mathbf{1}_{\{t \leq \varsigma\}} + z_\varsigma \mathbf{1}_{\{t > \varsigma\}} = z_t \mathbf{1}_{\{t \leq \varsigma\}} + \alpha \mathbf{1}_{\{t > \varsigma\}}.$$

Thus  $\alpha - z_t^\varsigma = (\alpha - z_t) \mathbf{1}_{\{t \leq \varsigma\}} \geq 0$  and  $E[\alpha - z_t^\varsigma] = \alpha - E[z_t^\varsigma] = \alpha$  for all  $t \in [0, T]$ . However,  $E[\alpha - z_T^\varsigma] = 0$ . Evidently,  $\{z_t^\varsigma : t \in [0, T]\}$  is a martingale that fails to be uniformly integrable.

Consider any increasing sequence of stopping times  $\{\tau(n)\}_{n=1}^\infty$ , where  $\tau(n) \in (0, T)$  and  $\tau(n) \rightarrow T$  almost surely. Then,

$$\lim_{n \rightarrow \infty} P(\{x : |\alpha - z_{\tau(n)}^\varsigma(x)| > \varepsilon\}) = 0 \quad \forall \varepsilon > 0.$$

On the other hand, we have

$$\lim_{n \rightarrow \infty} E[|\alpha - z_{\tau(n)}^\varsigma|] = \alpha.$$

In other words,  $\{z_{\tau(n)}^\varsigma\}_{n=1}^\infty$  converges in probability to  $\alpha$ ,<sup>38</sup> but it does not converge in the  $L_1(P)$  norm. Thus, the trading strategy does not generate an arbitrage because it is inadmissible with respect to the  $L_1(P)$  norm topology.

Regarding the weak-\* topology, there are two possibilities: either  $\{z_{\tau(n)}^\varsigma\}_{n=1}^\infty$  converges or it does not. If not, then it does not generate an arbitrage because it is inadmissible. On the other hand, if (as we suspect) it does converge, then the value of the limit point is zero and consequently the limit point has a strictly negative component; therefore, the trading strategy does not generate an arbitrage in this case either.<sup>39</sup>

The difference between the doubling strategy as described in this section and the trading strategy that generates an A-D security as described in the preceding section is this: We cannot express  $z_t^\varsigma = \int_0^t a_u dS_u$  in terms of a function  $g(S_t, t)$ .

The question is this: What does  $T_P^{-1}(z_{\tau(n)}^\varsigma)$  look like?

<sup>38</sup>In fact, it converges almost surely to  $\alpha$ .

<sup>39</sup>If  $\{z_{\tau(n)}^\varsigma\}_{n=1}^\infty$  converges in the weak-\* topology, then  $T_P^{-1}(z_{\tau(n)}^\varsigma) \xrightarrow{w^*} P + \mu$  for some  $\mu \in B_P^d$ . However,  $\langle \mathbf{1}, P + \mu \rangle = 0 \implies P + \mu \notin M(X)^{++}$ .

## APPENDIX A. THE CANTOR SPACE

The Cantor space  $\mathcal{C} = \{0, 1\}^\infty$  of infinite sequences of zeros and ones is compact in the product topology (where  $\{0, 1\}$  is equipped with the discrete topology).<sup>40</sup> The topology is metrizable.<sup>41</sup> In fact, the Cantor space is the mother of all compact metrizable spaces: Every compact metrizable space is homeomorphic to a quotient space of the Cantor space.<sup>42</sup> We illustrate this with two examples.

First, consider the following equivalence relation:  $x \sim y$  if

$$\sum_{i=1}^{\infty} 2^{-i} x(i) = \sum_{i=1}^{\infty} 2^{-i} y(i).$$

For example,  $\{1, 0, 0, \dots\} \sim \{0, 1, 1, \dots\}$ . In this case,  $\mathcal{C}/\sim$  is homeomorphic to the closed unit interval on the real line  $[0, 1]$  via the homeomorphism  $p : \mathcal{C}/\sim \rightarrow [0, 1]$  where  $p([x]) = \sum_{i=1}^{\infty} 2^{-i} x(i)$ .

Second, consider the following. For  $n \in \mathbb{N}$  and  $0 \leq i \leq 2^n - 1$ , define

$$B_n^i := \left\{ x \in \mathcal{C} : \sum_{j=1}^n 2^{n-j} x(j) = i \right\}. \quad (\text{A.1})$$

Let  $B_0^0 := \mathcal{C}$  and  $B_\infty^1 := \{0, 0, \dots\}$ . Note  $\{B_n^1\}_{n \in \mathbb{N}_\infty}$  is a partition of  $\mathcal{C}$ . (If  $x \in B_n^1$ , then  $x(n) = 1$  and  $x(i) = 0$  for  $i < n$ .) Define the equivalence relation by  $x \sim y$  if  $x, y \in B_n^1$ . In this case,  $\mathcal{C}/\sim$  is homeomorphic to  $\mathbb{N}_\infty$  via the homeomorphism  $p : \mathcal{C}/\sim \rightarrow \mathbb{N}_\infty$ , where  $p([x \in B_n^1]) = n$ .

**More on  $\mathcal{C}$ .** The reference measure is characterized by  $\eta(B_n^i) = 2^{-n}$  and  $\eta(\mathcal{C}) = 1$ .<sup>43</sup>

Let  $(\mathcal{C}, \mathcal{B}(\mathcal{C}), \varphi)$  be a probability measure space where  $\varphi(B_n^i) = \alpha_n^i$ . Let  $\alpha_n := \alpha_n^1$ . Note  $\varphi(\mathcal{C}) = 1$  and  $\text{supp } \varphi = \mathcal{C}$ . Similarly, we have  $\lambda(B_n^i) = \beta_n^i$ . Given the measurable space  $(\mathcal{C}, \mathcal{B}(\mathcal{C}))$ , a random variable is a measurable function  $Z : \mathcal{C} \rightarrow \mathbb{R}$ .

<sup>40</sup>The Cantor set (as it is usually defined) is the following set of points in the unit interval:

$$C = \left\{ \sum_{i=1}^{\infty} 3^{-i} a(i) : a(i) = 0 \text{ or } a(i) = 2 \right\}.$$

The Cantor space  $D$  is homeomorphic to the Cantor set  $C$  via the homeomorphism  $g : \mathcal{C} \rightarrow [0, 1]$ , where  $g(x) = \sum_{i=1}^{\infty} 3^{-i} 2x(i)$ .

<sup>41</sup>A metric that generates the product topology  $\tau_D$  is

$$d(x, y) = \sum_{i=1}^{\infty} 3^{-i} |x(i) - y(i)|,$$

for  $x, y \in \mathcal{C}$ . Thus  $(\mathcal{C}, d)$  is a compact metric space and  $x_n \xrightarrow{\tau_D} x \iff d(x_n, x) \rightarrow 0$ .

<sup>42</sup>Let  $X$  be a topological space and let  $\sim$  be an equivalence relation on  $X$ . Let  $X/\sim$  denote the set of all equivalence classes  $[x] = \{y \in X : x \sim y\}$ .  $X/\sim$  is called the quotient space of  $X$  by the equivalence relation  $\sim$ . Define a function  $q : X \rightarrow X/\sim$  by  $q(x) = [x]$ . This map  $q$  is called the quotient map. Define a set  $A \subset X/\sim$  to be open if  $q^{-1}(A)$  is open in  $X$ . This collection of open sets defines a topology on  $X/\sim$  called the quotient topology. If  $X$  is compact, then so is  $X/\sim$ . A function  $f : X/\sim \rightarrow Y$  is continuous if and only if the composite function  $f \circ q : X \rightarrow Y$  is continuous.

<sup>43</sup>Does  $\{B_n^i\}_{i=1}^{2^n-1}$  for  $n \in \mathbb{N} \cup \{\infty\}$  constitute a semi-ring?

Note  $T_\varphi^{-1}(Z) \in B_\varphi \subset M(\mathcal{C})$ . The behavior of a random variable  $Z$  on  $\varphi$ -null sets is of no consequence. Any countable collection of elements of  $\mathcal{C}$  is  $\varphi$ -null.

A point mass is  $\delta_x \in B_\varphi^d \subset M(\mathcal{C})$  where  $x \in \mathcal{C}$  is an infinite sequence of zeros and ones. Consider the countable set  $\mathcal{X} = \{x \in \mathcal{C} : \lim_{n \rightarrow \infty} x(n) \text{ exists}\}$ . Then  $\{\delta_x\}_{x \in \mathcal{X}}$  is dense in  $M(\mathcal{C})$ , since  $\mathcal{X}$  is dense in  $\mathcal{C}$ . We can approximate  $\varphi$  (and  $\lambda$ ) via  $\{\delta_x\}_{x \in \mathcal{X}}$ .

How about duality and integration:

$$\langle f, \mu \rangle = \int_{\mathcal{C}} f d\mu$$

and

$$\int_{\mathcal{C}} z d\lambda = E^\lambda[z].$$

To construct a filtration, let

$$\mathcal{F}_n := \sigma(\{B_n^i\}_{i=0}^{2^n-1}).$$

An adapted stochastic process is an infinite sequence of random variables  $\mathcal{Z} = \{Z_i\}_{i=1}^\infty$  where  $Z_i$  is  $\mathcal{F}_i$ -measurable. We can define conditional probabilities:

$$p_n^i = \begin{cases} \alpha_n^i / \alpha_{n-1}^{i/2} & i \text{ even} \\ \alpha_n^i / \alpha_{n-1}^{(i-1)/2} & i \text{ odd.} \end{cases}$$

For example, if  $p_n^i = \rho$ , then  $\varphi(B_n^i) = \rho^k (1 - \rho)^{n-k}$  given  $k = \sum_{j=1}^n x(j)$ .

Note  $B_n^{2^i} \cap B_n^{2^{i+1}} = \emptyset$  and  $B_n^{2^i} \cup B_n^{2^{i+1}} = B_{n-1}^i$  for  $0 \leq i \leq 2^{n-1} - 1$ . Therefore,  $\alpha_n^{2^i} + \alpha_n^{2^{i+1}} = \alpha_{n-1}^i$  and consequently  $p_n^{2^i} + p_n^{2^{i+1}} = 1$ . (Conditional on  $B_{n-1}^i$ , the probability of the next element being 0 is  $p_n^{2^i}$  while the probability of it being 1 is  $p_n^{2^{i+1}} = 1 - p_n^{2^i}$ .)

Define

$$\zeta_n^i := \begin{cases} -\sqrt{(1 - p_n^i)/p_n^i} & i \text{ even} \\ \sqrt{(1 - p_n^i)/p_n^i} & i \text{ odd.} \end{cases}$$

Note  $\zeta_n^{2^i} = -(\zeta_n^{2^{i+1}})^{-1}$ . Let

$$\begin{aligned} u_n^{2^i} &= u_{n-1}^i + \zeta_n^{2^i} \\ u_n^{2^{i+1}} &= u_{n-1}^i + \zeta_n^{2^{i+1}}, \end{aligned}$$

subject to  $u_0^0 = 0$ . Note

$$u_n^i = \sum_{j=1}^n \zeta_j^{[2^{j-n} i]},$$

where  $[x]$  denotes the floor of  $x$ . Define  $e_n^i := \mathbf{1}_{B_n^i}$ . Let  $U_n := \sum_{i=0}^{2^n-1} u_n^i e_n^i$ . Then

$$\Delta U_n := U_n - U_{n-1} = \sum_{i=0}^{2^n-1} \zeta_n^i e_n^i.$$

Let  $E_{n-1}^\varphi[Z] := E^\varphi[Z | \mathcal{F}_{n-1}]$ . Note

$$E_{n-1}^\varphi[\Delta U_n](x \in B_{n-1}^i) = \zeta_n^{2^i} p_n^{2^i} + \zeta_n^{2^{i+1}} p_n^{2^{i+1}} = 0$$

and

$$E_{n-1}^\varphi[(\Delta U_n)^2](x \in B_{n-1}^i) = (\zeta_n^{2i})^2 p_n^{2i} + (\zeta_n^{2i+1})^2 p_n^{2i+1} = 1.$$

We can write

$$\Delta S_n = m_n + \sigma_n \Delta U_n$$

where  $m_n$  and  $\sigma_n$  are  $\mathcal{F}_{n-1}$  measurable. (We assume  $\sigma_n \neq 0$ ). For  $x \in B_{n-1}^i$ , let  $m_n(x) = m_n^i$  and  $\sigma_n(x) = \sigma_n^i$ . Then

$$E_{n-1}^\varphi[\Delta S_n] = \sum_{i=0}^{2^{n-1}-1} m_n^i e_{n-1}^i$$

and

$$E_{n-1}^\varphi[(\Delta S_n - m_n)^2] = \sum_{i=0}^{2^{n-1}-1} (\sigma_n^i)^2 e_{n-1}^i.$$

Moreover,

$$S_n = S_0 + \sum_{j=1}^n \Delta S_j = S_0 + \sum_{j=1}^n m_j e_j + \sum_{j=1}^n \sigma_j \zeta_j e_j.$$

Define  $\ell_n := m_n/\sigma_n$ . Let  $X_n := 1 - \ell_n \Delta U_n$  and  $Y_n := \prod_{i=1}^n X_i$ . Note  $\Delta Y_n = -Y_{n-1} \ell_n \Delta U_n$ . Thus  $E_{n-1}^\varphi[X_n] = 1$  and  $E_{n-1}^\varphi[\Delta Y_n] = 0$ .

Define  $q_n^i := (1 - \ell_n^i \zeta_n^i) p_n^i$ . Then we can write

$$X_n = \sum_{i=0}^{2^n-1} \left( \frac{q_n^i}{p_n^i} \right) e_n^i$$

and (?)

$$Y_n = \sum_{i=0}^{2^n-1} \left( \frac{\prod_j q_j^k}{\prod_j p_j^k} \right) e_n^i.$$

Note

$$B_n^1 = \bigcup_{j=2^i}^{2^{i+1}-1} B_{n+i}^j.$$

and therefore

$$B_{n-i}^1 = \bigcup_{j=2^i}^{2^{i+1}-1} B_n^j.$$

Note  $(\bigcup_{i=1}^{n-1} B_i^1) \cup B_n^1 \cup B_n^0 = \mathcal{C}$ .

Let  $e_n := e_n^1$  and  $\tilde{e}_n := e_n^0$ . Then  $\mathbf{1} = \sum_{i=1}^n e_i + \tilde{e}_n$ .

In addition,  $\Delta G_n = \phi_n \Delta S_n = \phi_n m_n + \phi_n \sigma_n \Delta U_n$  where  $\phi_n$  is  $\mathcal{F}$ -measurable. Conditional on  $B_{n-1}^i$ , we have

$$\begin{aligned} G_n^{2i} &= G_{n-1}^i + \phi_n^i (m_n^i + \sigma_n^i \zeta_n^{2i}) \\ G_n^{2i+1} &= G_{n-1}^i + \phi_n^i (m_n^i + \sigma_n^i \zeta_n^{2i+1}). \end{aligned}$$

Given  $i = 0$  and fixing  $G_n^1$  we can solve for

$$\phi_n^0 = \frac{G_n^1 - G_{n-1}^0}{m_n^0 + \sigma_n^0 \zeta_n^1}. \quad (\text{A.2a})$$

$$G_n^0 = \frac{G_{n-1}^0 \sigma_n^0 (\zeta_n^1 - \zeta_n^0) + G_0^n (m_n^0 + \sigma_n^0 \zeta_n^0)}{m_n^0 + \sigma_n^0 \zeta_n^1} \quad (\text{A.2b})$$

Here is the doubling strategy. Let  $G_0 = 0$  and  $G_n^1 = 1$  for all  $n \in \mathbb{N}$ . For the classic doubling strategy, let  $m_n^0 = 0$ ,  $\sigma_n^0 = 1$ , and  $p_n^0 = p_n^1 = \frac{1}{2}$  so that  $-\zeta_n^0 = \zeta_n^1 = 1$ . Then  $G_n^0 = 1 - 2^n$  and  $\phi_n^0 = 2^{n-1}$ .

Given the trading strategy  $G_0 = 0$  and  $\phi_n^i = (a - G_{n-1}^i)/(S_n^{2i+1} - S_{n-1}^i)$ , define the stopping time  $\tau = \inf\{n : G_n = a\}$ . Note  $\tau(x \in B_n^1) = n$  for  $n \in \mathbb{N}_\infty$ . Thus  $\tau < \infty$  ( $\varphi$ -a.s.).

Define  $z_n = \sum_{j=1}^n 2^{j/2} \Delta U_j$  and let  $p_n^i = 1/2$ . Then

$$\Delta U_n^i = \zeta_n^i = \begin{cases} -1 & i \text{ even} \\ 1 & i \text{ odd.} \end{cases}$$

Define the stopping time  $\tau = \inf\{n : z_n \geq 1\}$ . For  $x \in B_1^1$ ,  $\tau = 1$ . For  $n \geq 2$ ,

$$-\sum_{j=1}^{n-2} 2^{j/2} + \left(2^{(n-1)/2} + 2^{n/2}\right) = 2 + \sqrt{2} > 1.$$

Can we show that the sequence of implied measures converges? Can we show that the sequence does not converge?

## APPENDIX B. A MORE GENERAL SETTING

Here we sketch a more general setting than that described in Section 1.

Suppose  $X$  is a Lusin space; i.e.,  $X$  is homeomorphic to a Borel subset of a compact metric space. Let  $C_b(X)$  denote the space of continuous bounded functions on  $X$ . Let  $ca_r(\mathcal{B}(X))$  denote the space of regular countably additive finite signed measures on  $\mathcal{B}(X)$ , the Borel  $\sigma$ -algebra of  $X$ . Let  $U = \{\mu \in ca_r(\mathcal{B}(X)) : \|\mu\| \leq 1\}$  denote the unit ball in  $ca_r(\mathcal{B}(X))$ . By adapting the argument in Rogers and Williams (1994, pp. 200–210), it can be show that (1)  $U$  is compact in the weak-\* topology, (2) the weak-\* topology of  $U$  is metrizable, and (3) weak-\* convergence is characterized by

$$\mu_n \xrightarrow{w^*} \mu \iff \langle f, \mu_n \rangle \rightarrow \langle f, \mu \rangle \quad \forall f \in C_b(X),$$

where  $\{\mu_n\}_{n=1}^\infty \subset ca_r(\mathcal{B}(X))$  and  $\mu \in ca_r(\mathcal{B}(X))$ .

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