AROUND AND AROUND: THE EXPECTATIONS HYPOTHESIS

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ABSTRACT. We show how to construct models of the term structure of interest rates in which the expectations hypothesis holds. McCulloch (1993) presents such a model, thereby contradicting an assertion by Cox, Ingersoll, Jr., and Ross (1981), but his example is Gaussian and falls outside the class of finite-dimensional Markovian models. We generalize McCulloch's model in three ways: (*i*) We provide an arbitrage-free characterization of the unbiased expectations hypothesis in terms of forward rates; (*ii*) we extend this characterization to a whole class of expectations hypotheses; and (*iii*) we show how to construct finite-dimensional Markovian and non-Gaussian examples.

INTRODUCTION

In one form or another, the expectations hypothesis has played a central role in the analysis of the term structure of interest rates. Perhaps the most common form of the expectations hypothesis is the so-called unbiased expectations hypothesis (U–EH) that asserts that forward rates equal the conditional expectations of future spot rates, but other forms exist as well. Cox, Ingersoll, Jr., and Ross (1981, CIR) characterize a number of mutually incompatible forms of the expectations hypothesis besides the U–EH, including the local expectations hypothesis (L–EH), under which the expected rate of return on all zero-coupon bonds—on all assets, in fact—equals the short-term risk-free rate.[1] Of the various expectations hypotheses they consider, CIR claim that only the L–EH is consistent with general equilibrium in continuous-time models (as well as in discrete-time models with continuous compounding of yields).

McCulloch (1993) provides a counter-example to CIR's claim. His example is in the spirit of Heath, Jarrow, and Morton (1992, HJM), in the sense that it does not admit a representation in terms of a finite number of Markovian state variables, the setting in CIR. Indeed, McCulloch suggests that CIR's claim may be true within their setting. As we show, it is not.

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[1]See Jarrow (1981) for a similar analysis.

There is a weak version of the U–EH according to which forward rates are biased predictors of future spot rates, but the bias, or term premium, is a constant that only depends on the forecast horizon. A regression of future spot rates on current forward rates produces a slope coefficient equal to unity under both versions, but the intercept, which equals zero under the strong version, is unrestricted under the weak version.[2] To construct an arbitrage-free model of the yield curve in which the weak U–EH holds, it is sufficient to let all volatilities be constant—the Gaussian case.[3] By contrast, our objective is (in part) to construct models of the strong version. In the process, we show that the U–EH does not require the Gaussian assumption.

A statement of the central idea of the paper may be useful at this point, because technical details sometimes stand in the way of the intuition. The difference between a forward rate and the expected future short rate at the corresponding horizon is the sum of a risk premium and a Jensen inequality term, also called a convexity premium. For the U–EH—or any other version of the expectations hypothesis—to hold, these two premia must balance each other in just the right way at all horizons. To choreograph the steps that each term is allowed to take, we introduce a vector-valued function $\phi(t,T)$, where t is the current time and T is the maturity date (so that $\tau := T - t$ is the horizon); this function involves both the market price of risk and bond price volatility. As long as, for any fixed t, $\phi(t,T)$ lies on a sphere, the balance between the risk and the convexity premia is achieved and the expectations hypothesis holds. If, in addition, the orbit of $\phi(t,t+\tau)$ is a circle on the sphere around which $\phi(t,t+\tau)$, considered as a function of τ, proceeds at a constant speed, then the model is Markovian.

We start in Section I with an example of a general-equilibrium production economy that fits in the class analyzed in CIR (1985). In this economy, although the U–EH holds, the fact is not obvious. The remainder of the paper is devoted to showing what makes this example work and how to construct others. In Section II, we generalize McCulloch's example by putting the U–EH explicitly into the HJM framework, focusing on absence-of-arbitrage conditions rather than building from a general equilibrium model. Then, in Section III, we extend the analysis to a class of expectations hypotheses that includes as special cases those considered by CIR (1981). Finally, in Section IV, we provide a method to construct examples in which there is a finite-dimensional Markov state vector (so-called factor models). We show that there exists only one two-factor model that is Gaussian. But, as we also show, there exist many other models that have either random volatilities or more than two factors, and we provide some examples. Some of these models are members of the exponential-affine class of term structure models characterized by Duffie and Kan (1993).

[2]Campbell and Shiller (1991), for example, test, and reject, the weak version of the U–EH.
[3]See Campbell (1986) for a general-equilibrium example.

1. A general-equilibrium example of the U–EH

In the setting analyzed in CIR (1985), wealth and a finite number of Markov factors summarize the state of the economy. These factors follow continuous-time diffusion processes. The part of wealth that is not consumed is reinvested in a finite number of stochastic and linear technologies to provide for future consumption. A representative consumer with initial wealth $k(0)$ chooses the consumption and investment plan to maximize the expected value of the flow of utility.

In our example, instantaneous utility is logarithmic and the discount rate is constant, so that the consumer maximizes

$$E\left[\int_0^\infty e^{-\rho\, t}\,\log[c(t)]\,dt\right].$$

There are two factors, $r(t)$ and $x(t)$, two sources of risk represented by two orthonormal Brownian motions, $(W_1(t), W_2(t))$, and a single technology. The technology is described by a variable $n(t)$ which can be interpreted as the value at time t of one unit of consumption invested in the technology at time 0 and continuously reinvested. Assume that the law of motion for $n(t)$ is

$$\frac{dn(t)}{n(t)} = (r(t) + q^2\,\zeta^2(t))\,dt + q\,\zeta(t)dW_1(t).$$

As shown in CIR (1985), in this logarithmic economy the law of motion for marginal utility, $m(t) := \exp[-\rho\, t]/c(t)$, is the same as that for $n^{-1}(t)$. Therefore, Ito's lemma implies that

$$\frac{dm(t)}{m(t)} = -r(t)\,dt - q\,\zeta(t)dW_1(t).$$

In this representative-agent economy, marginal utility is the state-price deflator, so that the short risk-free rate is $r(t)$ and the market price of risk vector (one price per source of risk) is $\lambda(t) = (q\zeta(t), 0)$.[4] Finally, assume that the laws of motion for the factors are

$$dr(t) = x(t)\,dt + \omega\,\zeta(t)\,dW_2(t) \tag{1.1}$$

and

$$dx(t) = y(t)\,dt - \omega^2\,\zeta(t)\,dW_1(t), \tag{1.2}$$

The processes (1.1) and (1.2) are specified in greater generality than needed here, because they will serve as the basis for other examples. In the present context, let $\zeta(t) := z$ and $y(t) := (\bar{r} - r(t))$, and let q, z, ω, and $\bar{r}$ be constant.

It may not be obvious, but the U–EH holds in this economy when $q = 1$. Rather than proving this statement for our special case only, we conduct a general analysis of various versions of the expectations hypothesis, including the U–EH. This analysis begins within the framework of Heath, Jarrow, and Morton (1992), which does not involve Markov factors and focuses on the conditions for absence of arbitrage rather than on those for equilibrium. After obtaining a criterion for the expectations hypothesis in a model of the HJM type, we focus on the conditions that such a

[4] See Duffie (1996) for a discussion of the state-price deflator and its relationship to marginal utility.

model must satisfy to admit a CIR Markov representation. Then we simply verify that these conditions hold in the foregoing example.

2. The U–EH in an HJM setting

In this section, we generalize McCulloch's example by characterizing the U–EH in terms of the HJM absence-of-arbitrage restriction. We start with some notation.

Let $P(t,T)$ denote the price at time t of a default-free zero-coupon bond that pays one unit of account at time T. Assume that, at any time t, $P(t,T)$ is a differentiable function of T, and define $f(t,T) := -\frac{\partial}{\partial T}\log[P(t,T)]$, the instantaneous forward rate for horizon T; this definition, of course, implies

$$\log[P(t,T)] = -\int_{s=t}^{T} f(t,s)\,ds. \tag{2.1}$$

Following CIR, define the unbiased expectations hypothesis as the proposition that forward rates are the conditional expectation of future spot rates; that is, $f(t,T) = E_t[r(T)]$, where, by definition, the short rate at time t is $r(t) := \lim_{T\to t} f(t,T)$, and $E_t[\,\cdot\,]$ denotes conditional expectation.

In the HJM approach to modeling the term structure, the primitives are (i) an initial forward curve $\{f(0,t) \mid t > 0\}$, (ii) the process for the market price of risk $\lambda(t)$, and (iii) the volatility of forward rates. We restrict attention to economies in which forward rates are diffusions driven by a d-dimensional vector $W(t)$ of standard Brownian motions. Let the process for forward rates be

$$df(t,T) = \mu_f(t,T)\,dt + \sigma_f(t,T)^\top\,dW(t), \tag{2.2}$$

where $\sigma_f(t,T)$ is a d-dimensional vector of forward-rate volatilities. Note that the market price of risk, $\lambda(t)$, is also a d-dimensional vector, and that it could be a random process, as could $\mu_f(t,T)$ and $\sigma_f(t,T)$.

To study conditions under which the U–EH holds, we need the relationship between future short rates and current forward rates. Since $r(T) = f(T,T) = f(t,T) + \int_{s=t}^{T} df(s,T)$, we can write

$$E_t[r(T)] = f(t,T) + \int_{s=t}^{T} E_t[df(s,T)] = f(t,T) + \int_{s=t}^{T} E_t[\mu_f(s,T)]\,ds.$$

Define the forward rate premium as follows:

$$\psi(t,T) := f(t,T) - E_t[r(T)] = \int_{s=t}^{T} E_t[-\mu_f(s,T)]\,ds. \tag{2.3}$$

Equation (2.3) clearly implies that if forward rates are unbiased predictors of future spot rates, then forward rates are martingales: $\mu_f(t,T) \equiv 0$. If, in addition, the ergodic distribution of $r(t)$ exists and has a mean, then the average yield curve is flat. By contrast, an upward sloping average yield curve requires an expected *decrease* in forward rates on average.

It might seem easy, then, to construct examples of the U–EH by choosing processes for forward rates (2.2) with $\mu_f(t,T) \equiv 0$. The problem is that doing so arbitrarily might introduce arbitrage opportunities. The condition for absence of arbitrage in the HJM setting specifies the drift of forward rates as[5]

$$\mu_f(t,T) = \sigma_f(t,T)^\top \left(\lambda(t) + \int_{s=t}^{T} \sigma_f(t,s)\,ds\right) \tag{2.4}$$

for all $0 \le t \le T$. Equation (2.4) shows that we must be able to write forward rate drifts in terms of their volatilities and the market price of risk for bond prices to be free of arbitrage opportunities. Since the U–EH requires forward rates to be martingales, the HJM characterization of this hypothesis is

$$\sigma_f(t,T)^\top \left(\lambda(t) + \int_{s=t}^{T} \sigma_f(t,s)\,ds\right) = 0. \tag{2.5}$$

The modeling challenge, then, is to find processes $\{\lambda(t),\, \sigma_f(t,T)\}$ that satisfy (2.5).

To meet this challenge, it is convenient to define

$$\phi(t,T) := \lambda(t) + \int_{s=t}^{T} \sigma_f(t,s)\,ds. \tag{2.6}$$

Note that $\phi'(t,T) = \sigma_f(t,T)$, where we define $F'(t,T) := \frac{\partial}{\partial T} F(t,T)$ for any function $F(\,\cdot\,,\,\cdot\,)$. Using (2.6), we can write (2.5) as

$$\phi'(t,T)^\top \phi(t,T) = 0. \tag{2.7}$$

Any function $\phi(t,T)$ that satisfies (2.7) has constant length: $\|\phi(t,T)\| = \|\phi(t,t)\|$. Two comments are in order. First, note that when $d = 1$, this condition can be satisfied only if $\phi(t,T)$ is a constant function of its second argument, in which case $\sigma_f(t,T) \equiv 0$, which means there is no uncertainty. Thus in order for (2.7) to hold when interest rates are stochastic, there must be at least two Brownians. Second, note that (2.7) does not restrict how $\phi(t,T)$ behaves as a function of its first argument: In particular, $\phi(t,T)$ can be a stochastic process.

We can restate the key relationship between $\phi(t,T)$ and the U–EH as follows:

> If $\phi(t,T)$ is a *rotation* of $\phi(t,t)$, then the U–EH is satisfied in an arbitrage-free way.

This statement yields a simple recipe for constructing arbitrage-free models of the U–EH: (*i*) choose $\phi(t,T)$ such that $\phi(t,t)$ is some random process and $\phi(t,T)$, for $T > t$, is a rotation of $\phi(t,t)$; (*ii*) set $\lambda(t) = \phi(t,t)$, and (*iii*) set $\sigma_f(t,T) = \phi'(t,T)$.

McCulloch (1993) constructs an economy in which the U–EH holds.[6] In McCulloch's example there are two sources of risk, so that $d = 2$. He chooses[7]

$$\phi(t, t+\tau) = a\left(\sqrt{2\,e^{-\tau} - e^{-2\tau}},\ 1 - e^{-\tau}\right)^\top.$$

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[5]This was first shown by HJM; see also Duffie (1996), p. 151 or Hull (1993), p. 398–401. The form of our restriction differs from the form that Duffie and Hull give because in their presentations $\mu_f(t,T)$ is risk adjusted, while here it is not.

[6]Frachot and Lesne (1994) note that such an example could be constructed easily by exploiting equation (2.5).

[7]We reverse the order of McCulloch's Brownian motions for comparison with what follows. Note that $a = \eta\sqrt{g_0}$ in his notation.

Note that $\|\phi(t,t+\tau)\| = \|\phi(t,t)\| = a$. Clearly, as τ increases, $\phi(t,t+\tau)$ turns continuously from $\phi(t,t) = (a,0)^\top$ to $\phi(t,t+\infty) = (0,a)^\top$, going a quarter of the way around the circle over the infinite horizon. Finally note that McCulloch's example is Gaussian since $\phi(t,T)$ is deterministic.

3. A class of expectations hypotheses

We now generalize the results from the previous section to encompass an entire class of expectations hypotheses. For this purpose, we will need to refer to the process for zero-coupon bonds:

$$\frac{dP(t,T)}{P(t,T)} = \mu_P(t,T)\,dt + \sigma_P(t,T)^\top dW(t). \tag{3.1}$$

From (2.1), note the following relation between the volatility of bond prices and that of forward rates

$$\sigma_P(t,T) = -\int_{s=t}^{T} \sigma_f(t,s)\,ds. \tag{3.2}$$

CIR characterize four versions of the expectations hypotheses: the U–EH, the L–EH, the Yield-to-Maturity Expectations Hypothesis (YTM–EH), and the Return-to-Maturity Expectations Hypothesis (RTM–EH). They show that the U–EH and the YTM–EH are identical in continuous time, and that—after imposing absence-of-arbitrage conditions—the three independent expectations hypotheses could be characterized in the following way:

$$\sigma_P(t,T)^\top \lambda(t) = \frac{q}{2}\,\|\sigma_P(t,T)\|^2, \tag{3.3}$$

where

$$q = \begin{cases} 0 & \text{under L–EH,} \\ 1 & \text{under YTM/U–EH, and} \\ 2 & \text{under RTM–EH.} \end{cases}$$

Equation (3.3) provides a characterization of the three expectations hypotheses. Moreover, it shows that they are mutually inconsistent unless $\sigma_P(t,T) \equiv 0$. Although CIR only consider $q \in \{0,1,2\}$, we allow q to be an arbitrary real number, and we refer to (3.3) as the q–expectations hypothesis (q–EH). CIR refer to (3.3) in making their claim that only the L–EH could hold in a continuous-time general equilibrium model. Clearly, the L–EH has a special status, since $q = 0$ implies $\lambda(t)$ is orthogonal to $\sigma_P(t,T)$ but imposes no other restriction; with $\lambda(t) \equiv 0$, for example, the L–EH is always satisfied and $\sigma_P(t,T)$ is unrestricted. For any other value of q, by contrast, if $\lambda(t)$ is orthogonal to $\sigma_P(t,T)$, then $\sigma_P(t,T) = 0$.

The left side of equation (3.3) is the excess expected rate of return over the next instant of time associated with holding the discount bond maturing at T, and it is therefore a risk premium. With $q = 1$, the right side is a Jensen's equality term, or convexity premium, that pulls the yield below the corresponding expected future short rate. When the two terms exactly equal each other, the two premia exactly offset each other, and the U–EH holds. Even with $q \neq 1$, the q–EH hypothesis requires a balance between the two premia, although not equality.

We can recast (3.3) in terms of forward rates by differentiating both sides with respect to T, using (3.2), and rearranging:

$$\sigma_f(t,T)^\top \left(\lambda(t) + q \int_{s=t}^{T} \sigma_f(t,s)\, ds \right) = 0. \tag{3.4}$$

We see that (2.5) is a special case of (3.4) with $q = 1$. It is convenient to generalize the definition of $\phi(t,T)$: Define $\phi(t,T)$ implicitly by

$$\lambda(t) = q\, \phi(t,t) \tag{3.5}$$

and

$$\sigma_f(t,T) = \phi'(t,T), \tag{3.6}$$

so that $q\, \phi(t,T) = \lambda(t) + q \int_{s=t}^{T} \sigma_f(t,s)\, ds$.

Using (3.5) and (3.6), we can write (3.4) as

$$q\, \phi'(t,T)^\top \phi(t,T) = 0. \tag{3.7}$$

Equation (3.7) is satisfied automatically if $q = 0$. If $q \neq 0$, (3.7) reduces to (2.7), in which case the comments that follow (2.7) apply here to the generalized definition of $\phi(t,T)$.

The recipe for constructing arbitrage-free models of the q–EH is this: (i) choose $\phi(t,T)$ such that $\phi(t,t)$ is some random process and $\phi(t,T)$, for $T > t$, is a rotation of $\phi(t,t)$; (ii) define $\lambda(t) = q\, \phi(t,t)$, and (iii) define $\sigma_f(t,T) = \phi'(t,T)$. For example, with McCulloch's $\phi(t,T)$, we could choose $\lambda(t) = q\, \phi(t,t)$ for *any* q.

Finally, note that we can restate the q–EH in terms of either forward rate drifts or term premia. Using (3.5) and (3.6), we can rewrite the no-arbitrage condition (2.4) as

$$\mu_f(t,T) = \phi'(t,T)^\top \Big((q-1)\, \phi(t,t) + \phi(t,T) \Big).$$

For $q \neq 0$, (3.7) implies the following characterization in terms of drifts:

$$\mu_f(t,T) = (q-1)\, \phi'(t,T)^\top \phi(t,t). \tag{3.8}$$

In view of (2.3), this equation holds if and only if

$$\psi(t,T) = (1-q) \int_{s=t}^{T} E_t[\phi'(s,T)^\top \phi(s,s)]\, ds, \tag{3.9}$$

which is a characterization of the q–EH in terms of the term premium. In what follows, we assume for convenience that (2.7), (3.8), and (3.9) hold even when $q = 0$.

4. Markovian models

McCulloch (1993) establishes decisively that the unbiased expectations hypothesis is consistent with general equilibrium, but he leaves open the possibility that expectations hypotheses may be inconsistent with general equilibrium in the finite-state Markovian world analyzed by CIR. We settle this issue by exhibiting Markov economies with two and three factors. The construction allows for the volatility of bond prices to be stochastic. The trick is to make $\phi(t,T)$ proceed around a circle at a constant pace, producing an infinite number of cycles.

In all of the examples we develop below, the processes for the short rate $r(t)$ and its drift $x(t)$ share the structure set out in (1.1) and 1.2, for various specifications of $\zeta(t)$ and $y(t)$.

To facilitate the analysis of the examples, we define a pair of functions that we will use repeatedly and for which $(q, \omega, \bar{r}, z)$ is a vector of fixed parameters:

$$Y(r,\zeta) := \omega^2 \left(\bar{r} - r + (q-1)\left(z^2 - \zeta^2\right)\right)$$

and

$$F_q(r,x,\tau) := \bar{r} + (r-\bar{r})\cos[\omega\,\tau] + x\,\frac{\sin[\omega\,\tau]}{\omega} + (q-1)\,z^2\,(1-\cos[\omega\,\tau]).$$

A two-factor model. Consider the following example, where $d = 2$ and $\phi(t,T)$ has constant norm z and turns at constant angular velocity ω:

$$\phi(t,t+\tau) = z\,C(\omega,\tau)\,, \tag{4.1}$$

where

$$C(\omega,\tau) = \begin{pmatrix}\cos[\omega\,\tau]\\ \sin[\omega\,\tau]\end{pmatrix}.$$

We prove in Proposition 2 below that this choice for $\phi(t,T)$ leads to processes for the short rate $r(t)$ and its drift $x(t)$ of the form (1.1) and (1.2), where $\zeta(t) = z$ and

$$y(t) = Y(r(t),z) = \omega^2\,(\bar{r} - r(t)).$$

In this model of the yield curve, the bivariate process for the two factors, $r(t)$ and $x(t)$, is Markovian. The short rate is stationary, with an unconditional mean equal to $\bar{r}$, while its drift $x(t)$ is also stationary, with an unconditional mean equal to zero; both unconditional variances are infinite. Volatilities are constant, so that the model is Gaussian. Finally, the market price of risk is

$$\lambda(t) = q\,\phi(t,t) = \begin{pmatrix} q\,z\\ 0\end{pmatrix}.$$

With linear drifts, constant volatilities, and a constant price of risk, the model belongs to the exponential-affine class introduced by Duffie and Kan (1995). The solution for forward rates is

$$f(t,t+\tau) = F_q(r(t),\,x(t),\,\tau). \tag{4.2}$$

At any time t, then, the forward curve is a sine wave with fixed angular frequency ω. At horizon $\tau = 0$, its level is $r(t)$ and its slope is $x(t)$, both random variables, so that the forward curve has random amplitude and phase shift. Using the methods described in Fisher and Gilles (1996), it is possible to verify that the conditional expectation of the short rate is

$$E_t[r(t+\tau)] = F_1(r(t),\,x(t),\,\tau), \tag{4.3}$$

while the conditional variance is $\mathrm{Var}_t[r(t+\tau)] = \omega^2\,z^2\,\tau$, converging linearly to infinity, the variance of the ergodic distribution. Clearly, the conditional mean and variance of $r(t)$ are independent of the value of q, as they must be since the process for $(r(t),\,x(t))$ is independent of q. But the value of q affects the market price of risk, and therefore the shape of the yield curve given in (4.2). These equations, imply that the term premium is

$$\psi(t,\, t+\tau) := f(t,\, t+\tau) - E_t[r(t+\tau)] = (q-1)\, z^2\, (1 - \cos[\omega\,(\tau)]),$$

which agrees with the term premium under the q–EH as given in equation (3.9). In particular, under the unbiased expectations hypothesis ($q = 1$) all term premia vanish.

We can now return to the general-equilibrium economy of Section 1. The market price of risk is $\lambda(t) = q(z, 0)$ while the short rate is $r(t)$. The yield curve is therefore exactly that of the example in this section, implying that the q–EH hypothesis holds in that economy; in particular, the U–EH holds if $q = 1$.

The example is the canonical Gaussian model. In the foregoing example, the factors have deterministic volatilities—the Gaussian case. From an empirical standpoint, Non-Gaussian models are more interesting, because they alone imply that term premia fluctuate randomly. Below, we construct such examples by generalizing the canonical example.

Before turning to the issue of non-Gaussian models, however, we prove two results about the canonical example, which clearly show that it is the natural place to start generalizing from. First, we show that there exists no one-factor model of the q–EH, Gaussian or not. This is simply because, under the q–EH, the univariate process for the short rate cannot be Markovian (all proofs appear in the appendix).

Proposition 1. *If the q–EH holds and the short rate $r(t)$ is not deterministic, then its univariate process is not Markovian.*

Second, we show that any two-factor Gaussian model of the q–EH is a renormalization of the canonical example.

Proposition 2. *Suppose that the q–EH holds in a model with two Markovian state variables such that bond prices have deterministic volatilities. Then there exist constant scalars $\bar{r}$, ω, and z such that, perhaps after changing the basis for the vector of Brownian motions (thus affecting the representation of the processes, but not the form of the yield curve):*

- *$\phi(t, T)$ has the form shown in (4.1);*
- *the processes for the short rate and its drift have the form shown in equations (1.1) and (1.2), with $\zeta(t) = z$ and $y(t) = Y(r(t), z)$;*
- *and the initial forward curve has the form*

$$f(0, \tau) = F_q(r(0),\, x(0),\, \tau). \tag{4.4}$$

Proposition 2 asserts that in a Gaussian and Markovian economy (with two factors), the q–EH implies that $\phi(t, T)$ keeps turning around the circle at constant angular velocity, ω. It also specifies the Brownian motion driving $x(t)$, the drift of $r(t)$, as orthogonal to that driving $r(t)$, and it specifies $y(t)$, the drift of $x(t)$, as a translation of $r(t)$ and independent of the value of $x(t)$ itself. The short rate is stationary with unconditional mean equal to $\bar{r}$, while its drift is also stationary with unconditional mean equal to zero.

Because the model is Markovian, the initial time has no particular significance, and equation (4.4) for the initial forward curve delivers the form of the generic forward curve, (4.2), that follows from solving a Duffie-Kan model. But while the Duffie-Kan method requires solving a simultaneous system of three Ricatti differential equations, we obtain the initial forward curve in the proof of Proposition 2 by solving a single second-order differential equation.

Clearly, forward curves can be flat; in fact the forward curve is flat if and only if $r(t) = \bar{r} + (q-1)\,z^2$ and $x(t) = 0$, because then $F_q(\bar{r} + (q-1)\,z^2,\, 0,\, \tau) = \bar{r} + (q-1)\,z^2$. The short rate and its drift have an ergodic distribution, with mean $\bar{r}$ and zero. Since F_q is linear in these two variables, the whole forward curve has an ergodic distribution, and its average is obtained by setting $r(t)$ and $x(t)$ at their unconditional means, $\bar{r}$ and 0:

$$F_q(\bar{r},\, 0,\, \tau) = \bar{r} + (q-1)\,z^2\,(1 - \cos[\omega\,\tau]).$$

The average forward curve is thus the same as the flat forward curve under the U–EH ($q = 1$), but in other cases it is a sine wave.

Non-Gaussian models. We now turn to the non-Gaussian case. The simplest way to generalize the canonical Gaussian example is to suppose that $\phi(t,t)$—which is proportional to the market price of risk $\lambda(t)$—is an Ito process. To do this without increasing the number of factors, replace equation (4.1) by $\phi(t, t+\tau) = \zeta(t)C(\omega,\, \tau)$ where $\zeta(t)$ is some function of $r(t)$ and $x(t)$. As shown in Proposition 3, the result is a non-Gaussian two-factor Markov model of the yield curve in which the q–EH holds. Although this approach deliver closed-form expressions for bond prices, checking that the q-hypothesis holds may not be easy in practice, because we do not have closed-form expressions for the conditional forecasts of the factors. For this reason, we also introduce in Proposition 4 a non-Gaussian three-factor model, in which we know how to compute both bond prices and conditional forecasts.

Proposition 3. *Let $\phi(t, t+\tau) = \zeta(t)\,C(\omega, \tau)$, where $C(\omega, \tau)$ is as in (4.1) and $\zeta(t) = \zeta(r(t), x(t))$, for any function $\zeta(\cdot, \cdot)$ (with the restriction that the implied stochastic processes for $r(t)$ and $x(t)$ have a solution). Suppose also that $\zeta^2(t)$ has an unconditional mean z^2. Pick a constant $\bar{r}$ and initial conditions $r(0)$ and $x(0)$, and choose the following initial forward curve*

$$f(0, \tau) = F_q(r(0),\, x(0),\, \tau).$$

Then:

- *the resulting yield curve model is Markovian with two factors, $(r(t), x(t))$, as well as non-Gaussian if $\zeta(t)$ is random, and it satisfies the q-expectations hypothesis;*
- *the processes for the two factors have the form shown in equations (1.1) and (1.2), with $y(t) = Y(r(t),\, \zeta(t))$;*
- *at any time t, the forward curve is*

$$f(t, t+\tau) = F_q(r(t),\, x(t),\, \tau). \tag{4.5}$$

We see that bond prices are independent of $\zeta(t)$ and depend on the other two factors, $r(t)$ and $x(t)$, exactly as they do in the corresponding Gaussian model. The only difference between the Gaussian and the non-Gaussian models is the distribution of these factors; therefore yield curves of a given shape do not occur with the same frequency in both models.

In the non-Gaussian model, the drift of $x(t)$ depends on $\zeta^2(t)$ (except when $q = 1$), which complicates the task of making conditional forecasts. If $\zeta^2(t)$ were a linear function of $x(t)$ and $r(t)$, then the model would be in the exponential-affine class, and we would know how to compute conditional forecasts. Unfortunately, in our two-factor model there is no guarantee that either the interest rate or its drift can stay positive (in fact, the mean of $x(t)$ equals zero), and no linear combination of these variables is guaranteed to stay positive. Therefore, $\zeta^2(t)$ cannot be a linear function of $(r(t), x(t))$. We can get around this problem by treating $\zeta^2(t)$ as a third, independent factor.

The following three-factor model belongs to the exponential-affine class.

Proposition 4. *Set $d = 3$. Let $\phi(t, t+\tau) = \zeta(t)\, C^*(\omega, \tau)$, where C^* is the modification of the function C given in (4.1) obtained by adding a third component which identically equals zero; let the process for $\zeta^2(t)$ satisfy*

$$d\zeta^2(t) = k\,(z^2 - \zeta^2(t))\,dt + \zeta(t)\,dW_3(t),$$

and let $f(0,\tau) = F_q(r(0),\, x(0),\, \tau)$, so that the initial forward curve is as in Proposition 3. Then the conclusions of Proposition (3) hold, with the obvious modification that the state is the three-dimensional vector $(r(t), x(t), \zeta^2(t))$.

The bond price formulas in the two- and three-factor models are identical. The only difference is that, because the latter model belongs to the exponential-affine class, it is possible to obtain closed-form solutions for the first two conditional moments of all factors. The mean of the short rate, in particular, satisfies

$$\begin{aligned} E_t[r(t+\tau)] = F_1(r(t),\, x(t),\, \tau) + \\ \frac{(1-q)\,\omega^2}{k^2+\omega^2}\left(\zeta^2(t) - z^2\right)\left(e^{-k\tau} - \cos[\omega\tau] + \frac{k\sin[\omega\tau]}{\omega}\right). \end{aligned}$$

Subtracting the right side from F_q produces the forward premium ψ:

$$\psi(t, t+\tau) = (q-1)\left\{z^2(1-\cos[\omega\tau]) - A\left(e^{-k\tau} - \cos[\omega\tau] + \frac{k\,\sin[\omega\tau]}{\omega}\right)\right\},$$

where $A := \omega^2(\zeta^2(t) - z^2)/(k^2+\omega^2)$. It can be further verified that, because

$$E_t[\zeta(t+s)^2] = z^2 + e^{-ks}\,\left(\zeta(t)^2 - z^2\right),$$

the above expression for the term premium agrees with (3.9), which in the present case reduces to

$$\psi(t, t+\tau) = (q-1)\,\omega \int_{s=0}^{\tau} E_t[\zeta^2(t+s)]\sin[\omega(\tau - s)]\,ds.$$

5. Concluding remarks

We have shown that the expectations hypothesis is compatible with general equilibrium even in finite-dimensional Markovian settings. The models we have been able to construct, however, will not help to rehabilitate the expectations hypothesis. Rather, they show how implausible the hypothesis is, because our examples all share the same process for the short rate, which implies that the forecast of the short rate path is a sine wave with nondampening amplitude. Moreover, the unconditional variance of the short rate is infinite, an undesirable feature from an econometric standpoint.

In the HJM setting, yield curves and conditional moments for the short rate can look more reasonable. The short rate in McCulloch's example, for instance, has finite variance, while its forecast (given by the current forward curve) can have any shape. Note that McCulloch did not exhibit a yield curve. In fact, in the HJM setting, the initial forward curve, which McCulloch did not specify, can be chosen arbitrarily, although it affects the form of the yield curve for all time to come.[8] The expectations hypothesis imposes restrictions only on the dynamics of the yield curve. Given the initial forward curve and its dynamics, it is in principle possible to reconstruct future yield curves for any path of the set of Brownian motions. But because no finite set of variables summarizes the state of the economy, the problem of keeping track of the whole history becomes unwieldy very quickly, and we cannot say what a typical forward curve looks like in McCulloch's example.

At first blush, it may seem that the models exhibited here have the potential to represent the cyclical behavior of interest rates prior to the existence of the Federal Reserve. Unfortunately, the models cannot be made to reasonably approximate that sort of cyclical behavior. The problem is that the pre-Fed cycles are seasonal, so that their phase is deterministic, while the cycles in our models are subject to random phase shifts. In other words, there is no way to make summer (for example) be a high-rate season on average.

As a final observation, we suspect that no equilibrium model of the expectations hypothesis, Markovian or non-Markovian, can guarantee the non-negativity of the short rate. This is certainly true in McCulloch's example and all of our examples. Such a feature makes the expectations hypothesis a poor benchmark for nominal rates. The reason for the inability to keep the short rate positive is simple. If the short rate is to stay positive, its volatility must be small enough and *its drift must be positive* whenever its level is close to zero. But in all our examples, the drift of the short rate is independent of the short rate itself, and therefore will not always point in the right direction when the rate is small.

Appendix A. Proofs of Propositions

Preliminaries. To prove the propositions, we need the process for the short rate under the q–EH. In the HJM framework, the three model primitives—the initial forward curve, the market price of risk and the volatility of forward rates—are

[8]Initial forward curves in our examples are determined only by the condition that the model is Markovian, as the proofs of the propositions make clear.

guaranteed to deliver an arbitrage-free model of the term structure. The short rate is given by

$$r(t) = f(t,t) = f(0,t) + \int_{s=0}^{t} \mu_f(s,t)\,ds + \int_{s=0}^{t} \sigma_f(s,t)^\top dW(s), \tag{A.1}$$

where $\mu_f(t,T)$ is given by equation (2.4). Using (3.6) and imposing the q–EH condition (3.8), equation (A.1) becomes

$$r(t) = f(0,t) + (q-1)\int_{s=0}^{t} \phi'(s,t)^\top \phi(s,s)\,ds + \int_{s=0}^{t} \phi'(s,t)^\top dW(s); \tag{A.2}$$

from which it follows that the process for the short rate obeys

$$dr(t) = x(t)\,dt + \phi'(t,t)^\top dW(t), \tag{A.3}$$

where

$$x(t) = f'(0,t) + (q-1)\int_{s=0}^{t} \phi''(s,t)^\top \phi(s,s)\,ds + \int_{s=0}^{t} \phi''(s,t)^\top dW(s). \tag{A.4}$$

From this definition, the process for $x(t)$ clearly obeys

$$dx(t) = y(t)\,dt + \phi''(t,t)^\top dW(t), \tag{A.5}$$

where

$$\begin{aligned} y(t) = f''(0,t) &+ (q-1)\,\phi''(t,t)^\top \phi(t,t) + \\ &(q-1)\int_{s=0}^{t} \phi'''(s,t)^\top \phi(s,s)\,ds + \int_{s=0}^{t} \phi'''(s,t)^\top dW(s). \end{aligned} \tag{A.6}$$

Proof of Proposition 1. If $r(t)$ is Markovian, then its drift $x(t)$ must be a function of $r(t)$, say $x(t) = g(r(t))$, for some continuous function $g(\cdot)$. Fix a time s and consider a disturbance in the path of $W(t)$ by $\Delta W(s)$. For simplicity, we assume g to be differentiable. Then from equations (A.2) and (A.4), we see that the changes at time s in the short rate and its drift are

$$\Delta r(s) = \phi'(s,s)^\top \Delta W(s), \quad \text{and} \quad \Delta x(s) = \phi''(s,s)^\top \Delta W(s).$$

But for $\Delta W(s)$ small enough, we must also have $\Delta x(s) = g'(r(s))\,\Delta r(s)$. This equation must be satisfied for any value of $r(s)$, so that

$$\phi''(s,s) = g'(r(s))\,\phi'(s,s) \qquad \text{for all } s \le t.$$

Since g' is a scalar, this equation implies that $\phi''(s,s)$ is proportional to $\phi'(s,s)$. Unless $\phi' = \phi'' = 0$, which is the deterministic case, ϕ'' cannot be proportional to ϕ' because they must be orthogonal for ϕ to stay on the circle. **Q. E. D.**

Proof of Proposition 2. In a two-factor model, we can set $d = 2$. The univariate process for the short rate $r(t)$ is not Markovian because its drift $x(t)$ cannot be a function of $r(t)$. But in a two-factor model, the vector $(r(t), x(t))$ must be Markovian itself. Thus, we may take $r(t)$ and $x(t)$ as the two factors. This requires $y(t)$, the drift of $x(t)$, be a function of $r(t)$ and $x(t)$, say $y(t) = g(r(t), x(t))$ for some continuous function g.

We assume without loss of generality that $\phi(t,t)$ has a non-zero entry (if any) only in its first component, and we write $\phi(t,t) = (\zeta(t),\, 0)$. There is no loss of generality because, if $\phi(t,t) \neq 0$, then we can change the orthonormal basis of Brownian motions to $B(t) = (B_1(t), B_2(t))$, where $B_1(t) = \phi(t,t)^\top W(t)/\|\phi(t,t)\|$. This procedure amounts to choosing a particular Choleski decomposition of the noise in the economy. From now on, we assume that $W(t)$ is chosen to start with to coincide with the basis $B(t)$.

Before proceeding further, we re-parameterize the function ϕ using polar coordinates to enforce the restrictions that $\phi(t,t) = (\zeta(t),\, 0)$ and that $\phi(t,T)$ lies on a circle. Write

$$\phi(t,T) = \zeta(t) \begin{pmatrix} \cos[\theta(t,T)] \\ \sin[\theta(t,T)] \end{pmatrix},$$

where $\theta(t,T)$, the angle determining the position of $\phi(t,T)$ on the circle, satisfies $\theta(t,t) = 0$. Clearly, $\|\phi(t,T)\| = \|\phi(t,t)\| = |\zeta(t)|$. By differentiation, we get

$$\phi'(t,T) = \zeta(t)\,\theta'(t,T) \begin{pmatrix} -\sin[\theta(t,T)] \\ \cos[\theta(t,T)] \end{pmatrix}, \tag{A.7}$$

$$\phi''(t,t) = \zeta(t) \left\{ \theta''(t,T) \begin{pmatrix} -\sin[\theta(t,T)] \\ \cos[\theta(t,T)] \end{pmatrix} - \theta'(t,T)^2 \begin{pmatrix} \cos[\theta(t,T)] \\ \sin[\theta(t,T)] \end{pmatrix} \right\}, \tag{A.8}$$

and

$$\begin{aligned} \phi'''(t,T) = \zeta(t) \Bigg\{ & \theta'''(t,T) \begin{pmatrix} -\sin[\theta(t,T)] \\ \cos[\theta(t,T)] \end{pmatrix} - 3\,\theta''(t,T)\,\theta'(t,T) \begin{pmatrix} \cos[\theta(t,T)] \\ \sin[\theta(t,T)] \end{pmatrix} \\ & - \theta'(t,T)^3 \begin{pmatrix} -\sin[\theta(t,T)] \\ \cos[\theta(t,T)] \end{pmatrix} \Bigg\}. \end{aligned} \tag{A.9}$$

Note that $\phi(t,T)$ is deterministic; this follows directly from (A.7) because bond prices have non-random volatilities by assumption, so that $\phi'(t,T) = \sigma_f(t,T)$ is deterministic. We now turn to the implications of the q–EH for the form of g and $\phi(t,T)$. Assume that g is differentiable (with this assumption, we find that it is in fact linear), and denote by g_1 and g_2 its partial derivatives with respect to $r(t)$ and $x(t)$. We use the variation method used in the previous proof: fix a time t and disturb the path of $\{W(s) \mid s > 0\}$ at some point in the past $\tau < t$. Since $\phi(t,T)$ is deterministic, it is unaffected by this change; as a result, the short rate $r(t)$, its drift $x(t)$, and $y(t)$, the drift of $x(t)$, change by the amounts

$$\Delta r(t) = \phi'(\tau,t)\,\Delta W(\tau),$$
$$\Delta x(t) = \phi''(\tau,t)\,\Delta W(\tau),$$

and

$$\Delta y(t) = \phi'''(\tau, t)\, \Delta W(\tau).$$

But we must have also (for infinitesimal changes)

$$\Delta y(t) = g_1(r(t), x(t))\, \Delta r(t) + g_2(r(t), x(t))\, \Delta x(t). \tag{A.10}$$

This must hold for any value of $(r(t), x(t))$, and any τ. Hence, g_1 and g_2 must be constant, so that for some scalars a, b and c, we have

$$y(t) = g(r(t), x(t)) = a + b\, r(t) + c\, x(t). \tag{A.11}$$

Now, given the forms of $\Delta r(t)$, $\Delta x(t)$ and $\Delta y(t)$, equation (A.11) holds if and only if

$$\phi'''(t, T) = b\, \phi'(t, T) + c\, \phi''(t, T). \tag{A.12}$$

Substituting (A.7–A.9) into equation (A.12) allows us to conclude that that restriction will be violated unless, for some constant ω,

$$\theta'(t, T) = \omega\,; \quad b = -\omega^2; \quad \text{and} \quad c = 0.$$

Since $\theta(t, t) = 0$, the fact that $\theta' = \omega$ implies $\theta(t, T) = \omega(T - t)$. Since $\zeta(t)$ is not stochastic, it must be constant (otherwise, absolute time would enter as an independent argument of g, so that, strictly speaking, $(r(t), x(t))$ would not be Markovian); let z denote the constant value of $\zeta(t)$. This establishes the first two parts of the proposition. The constant a in (A.11) is arbitrary, so that the drift of $x(t)$ is $y(t) = g(r(t), x(t)) = a - \omega^2\, r(t)$; but we let $a = \bar{r}\, \omega^2$, and choose to parameterize g in terms of $\bar{r}$ and ω, rather than a and ω.

To finish the proof, we need to find the form of the initial forward curve. Given the form of $\phi(t, T)$, equations (A.2) and (A.6) imply that

$$y(t) = f''(0, t) + \omega^2 \Big(f(0, t) - r(t) - (q - 1) z^2 \Big),$$

from which we conclude that

$$a = \bar{r}\, \omega^2 = f''(0, t) - (q - 1)\, \omega^2\, z^2 + \omega^2\, f(0, t). \tag{A.13}$$

This is a second order ordinary differential equation, a solution of which requires two boundary conditions. These conditions come from the initial conditions for the state of the system, $r(0)$ and $x(0)$. Clearly, $f(0, t) = r(0)$ and from equation (A.4), we see that $f'(0, t) = x(0)$. The unique solution to (A.13) subject to these initial conditions is (4.4). **Q. E. D.**

Proof of Proposition 3. We start by proving the second statement about the processes of $r(t)$ and $x(t)$. The drift of $r(t)$ is $x(t)$ by definition. Given the assumed form of $\phi(t, T)$, the volatilities of $r(t)$ and $x(t)$ are $(0,\, \omega\, \zeta(t))$ and $(-\omega^2\, \zeta(t),\, 0)$, as stated. The drift of $x(t)$, $y(t)$, can be calculated from (A.6). Given (A.2) and the form of $\phi(t, T)$, we see that $y(t) = f''(0, t) - \omega^2 \Big(r(t) - f(0, t) + (q - 1)\, \zeta^2(t) \Big)$. The stated result follows from the form of the initial forward curve, which implies that $f''(0, t) + \omega^2 f(0, t) = \omega^2 (\bar{r} + (q - 1)\, z^2)$.

We now turn to the first and third statements. The fact that the model satisfies the q-expectations hypothesis is built into the form of $\phi(t, T)$. The form of the processes for the short rate and its drift clearly show that $(r(t),\, x(t))$ is a Markovian vector, since $\zeta(t)$ is assumed to be a deterministic function of this vector. If the model of the yield curve is Markovian, then given the factor values, it should not independently matter what value the time index t has. In particular, equation (A.6) must still hold for any reference date t replacing date 0 (and $t+\tau$ replacing t). This implies that for any t, $f''(t, t+\tau) + \omega^2 f(t, t+\tau) = \omega^2(\bar{r} + (q-1)z^2)$. The only solution of this differential equation (where t is fixed and the variable is τ) satisfying the boundary conditions $f(t,t) = r(t)$ and $f'(t,t) = x(t)$ is

$$f(t, t+\tau) = \Big(\bar{r} + (q-1)\, z^2\Big)\Big(1 - \cos[\omega\, \tau]\Big) + r(t)\, \cos[\omega\, \tau] + \frac{x(t)\, \sin[\omega\, \tau]}{\omega}.$$

Q. E. D.

Proof of Proposition 4. The proof is identical to that of the previous proposition. **Q. E. D.**

References

Campbell, J. Y. (1986). A defense of traditional hypotheses about the term structure of interest rates. *Journal of Finance 41*, 183–193.

Campbell, J. Y. and R. J. Shiller (1991). Yield spreads and interest rate movements: A bird's eye view. *Review of Economic Studies 58*, 495–514.

Cox, J. C., J. E. Ingersoll, Jr., and S. A. Ross (1981). A reexamination of traditional hypotheses about the term structure of interest rates. *Journal of Finance 36*, 321–346.

Cox, J. C., J. E. Ingersoll, Jr., and S. A. Ross (1985). A theory of the term structure of interest rates. *Econometrica 53*(2), 385–407.

Duffie, D. (1996). *Dynamic Asset Pricing Theory* (second ed.). Princeton, New Jersey: Princeton University Press.

Duffie, D. and R. Kan (1995). A yield-factor model of interest rates. Working paper, Graduate School of Business, Stanford University. Forthcomming: *Mathematical Finance*.

Fisher, M. and C. Gilles (1996). Estimating exponential-affine models of the term structure. Working paper, Board of Governors of the Federal Reserve System, Washington, DC.

Frachot, A. and J.-P. Lesne (1994). Expectations hypotheses and stochastic volatilities. Working paper, Banque de France.

Heath, D., R. Jarrow, and A. Morton (1992). Bond pricing and the term structure of interest rates: A new methodology. *Econometrica 60*, 77–105.

Hull, J. C. (1993). *Options, Futures, and other Derivative Securities* (second ed.). Englewood Cliffs, NJ: Prentice-Hall.

Jarrow, R. A. (1981). Liquidity premiums and the expectations hypothesis. *Journal of Banking and Finance 5*, 539–546.

McCulloch, J. H. (1993). A reexamination of traditional hypotheses about the term structure of interest rates: A comment. *Journal of Finance 48*(2), 779–789.

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