THE TERM STRUCTURE OF REPO SPREADS

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Abstract. We characterize the term structure of repo spreads in terms of forward prices and forward rates. The analysis builds on Duffie (1996b), who showed that expected future overnight repo spreads are capitalized into a security's price. We show how to calculate correctly the forward price of a bond on special; we derive an absence-of-arbitrage restriction on the drift of the process for forward repo term spreads in the spirit of Heath, Jarrow, and Morton (1992); and we characterize an extension of the unbiased expectations hypothesis in terms of that restriction. Finally, we describe the average behavior of the term structure of repo spreads for U.S. Treasury securities over the auction cycle, and we examine empirically the unbiased expectations hypothesis of the term structure of repo spreads.

1. Introduction

Repurchase-agreement transactions can be thought of as collateralized loans; the loan is then said to finance the collateral. For most publicly traded U.S. Treasury securities the financing rate in the repo market is the general-collateral (repo) rate, which can be thought of as the risk-free rate. In contrast, for some Treasury securities—typically recently issued securities—the financing rate is *lower* than the general collateral rate. These securities are said to be on special, and their financing rates are referred to as specific-collateral (repo) rates. The difference between the general-collateral rate and the specific-collateral rate is the repo spread.

For some purposes it is more useful to think about a repurchase-agreement transaction as composed of two parts: an outright sale at the spot price and an agreement to repurchase at the forward price. From this perspective, the repo rate is simply another way to quote the forward price. Since the forward price plays a central role in the analysis of repo rates, it is important to calculate it correctly. The standard way to calculate the forward price is not correct when applied to specific collateral because it ignores potential repo earnings.

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Duffie (1996b) shows that current and future overnight repo spreads are capitalized into the price of the security, because the repo spread represents a potential "dividend" that can be earned by whoever possesses the collateral. In the repo market, Treasury securities can be financed overnight and for longer terms. For a given security and a given term, the term repo spread is the difference between the general-collateral rate for that term and the specific-collateral rate for the same term. Thus for each security there is a term structure of repo spreads. Using the framework of Heath, Jarrow, and Morton (1992), (HJM hereafter) we derive the restrictions implied by the absence of arbitrage for the term structure of repo spreads that are analogous to the restrictions HJM derived for general collateral.

The unbiased expectations hypothesis (U–EH) can be applied to the term structure of repo spreads: Forward repo spreads are the conditional expectation of future spot repo spreads. The U–EH implies that forward repo spreads are martingales, a restriction on the drift of forward repo spreads. We characterize the U–EH in terms of the absence of arbitrage condition for forward repo spread.

In the final part of the paper, we describe the average behavior of the term structure of repo spreads for U.S. Treasury securities over the auction cycle, and we examine empirically the U–EH for the term structure of repo spreads.

2. Repo spreads

Overnight repo spreads. We begin with the framework developed by Duffie (1996b) for pricing securities with potential repo earnings. His framework is an arbitrage-free, complete market where the instantaneous risk-free rate is explicitly identified with the general-collateral rate. Let the overnight[1] general-collateral rate at time t be $r(t)$, and let the equivalent martingale measure associated with deflation by $\exp\left(\int_{u=0}^{t} r(u)\,du\right)$ be $\mathcal{Q}$.[2] Under $\mathcal{Q}$, the total expected rate of return—capital gains plus dividends—on all assets is $r(t)$.

Consider a zero-coupon bond that pays one unit when it matures at time T and that will never be on special. We refer to such a bond as general collateral. The price of general collateral, $P(t,T)$, is given by

$$P(t,T) = E_t^{\mathcal{Q}}\left[\psi(t,T)\right], \tag{2.1}$$

where $\psi(t,\tau) := \exp\left(-\int_{u=t}^{\tau} r(u)\,du\right)$. Of course (2.1) is a standard expression for the term structure of interest rates.[3] Note that, by the law of iterated expectations, we can also write $P(t,T) = E_t^{\mathcal{Q}}\left[\psi(t,\tau)P(\tau,T)\right]$.

Now consider bonds that have the potential to be on special; to distinguish among such bonds, we use the index α. For simplicity, we analyze here the case where bonds on special are all zero-coupon bonds.[4] Let T_α denote the maturity of bond α, and

[1] "Overnight" is meant to be suggestive rather than literal. Literally, $r(t)$ is a rate over an infinitesimal period.

[2] See Duffie (1996a) for a discussion of the equivalent martingale measure.

[3] See Duffie (1996a), p. 131.

[4] Appendix A extends the analysis to coupon bonds.

$P_\alpha(t)$ denote its price at time t. Finally, let the overnight special-collateral rate at time t on this bond be $r_\alpha(t)$, and the overnight repo spread be[5]

$$s_\alpha(t) := r(t) - r_\alpha(t).$$

Duffie (1996b) presents arbitrage-based arguments to show that $s_\alpha(t) \geq 0$.

When a bond is on special, its owner can earn a "dividend" at a rate equal to the repo spread, $s_\alpha(t)$. Under $\mathcal{Q}$, the expected rate of return on the bond (that is, the sum of the dividend rate and the expected rate of price appreciation) is, as noted above, $r(t)$, since this is the expected rate of return on every asset. The specific-collateral repo rate, $r_\alpha(t) = r(t) - s_\alpha(t)$, must therefore equal the expected rate of price appreciation on bond α. This in turn implies that $P_\alpha(t)$ has a representation analogous to (2.1), with the specific-collateral rate replacing the general-collateral rate. Letting $\psi_\alpha(t,\tau) := \exp\left(-\int_{u=t}^{\tau} r_\alpha(u)\, du\right)$, we have

$$P_\alpha(t) = E_t^{\mathcal{Q}}\left[\psi_\alpha(t, T_\alpha)\right] = E_t^{\mathcal{Q}}\left[\psi_\alpha(t,\tau) P_\alpha(\tau)\right]. \tag{2.2}$$

Define τ_α to be the minimum time beyond which it is certain there will be no further repo earnings. (For a bond that may be on special until maturity, $\tau_\alpha = T_\alpha$.) Given τ_α, we can write $P_\alpha(t) = E_t^{\mathcal{Q}}\left[\psi_\alpha(t,\tau_\alpha) P(\tau_\alpha, T_\alpha)\right]$.

A bond with potential repo earnings is worth more than a comparable bond with no potential repo earnings by the present value of the flow of earnings. Let $\pi_\alpha(t)$ denote the relative price premium on bond α, which matures at T_α:

$$\pi_\alpha(t) := \log\left(\frac{P_\alpha(t)}{P(t, T_\alpha)}\right). \tag{2.3}$$

Clearly $\pi_\alpha(t) \geq 0$, since $s_\alpha(t) \geq 0$.

Forward prices and term repo spreads. Term repo rates are simply reparameterizations of forward prices in terms of spot prices.[6] For general collateral, the standard expressions obtain. However, because of potential repo earnings for specific collateral, we must be careful to derive the forward price of specific collateral correctly.

Let $F(t,\tau,T)$ denote the forward price at time t of general collateral that matures at time T to be delivered at time τ, where $t \leq \tau \leq T$. By definition, the forward price makes the value of the forward contract zero; thus the forward price is defined as the value that solves

$$E_t^{\mathcal{Q}}\Big[\psi(t,\tau)\Big(P(\tau,T) - F(t,\tau,T)\Big)\Big] = 0. \tag{2.4}$$

Solving (2.4) produces the standard expression:

$$F(t,\tau,T) = \frac{P(t,T)}{P(t,\tau)}. \tag{2.5}$$

The general-collateral term repo rate, $r(t,\tau)$, is implicitly defined by

$$P(t,T)\, \exp(r(t,\tau) \times (\tau - t)) = F(t,\tau,T). \tag{2.6}$$

[5]For general collateral, $s_\alpha(t) = 0$ for all $t \leq T_\alpha$.

[6]In practice one observes spot prices and term repo rates, from which forward prices have to be computed.

Combining (2.5) and (2.6), we see that $r(t,\tau)$ is simply the yield to maturity on general collateral that matures at time τ:

$$r(t,\tau) = -\frac{\log(P(t,\tau))}{\tau - t}. \tag{2.7}$$

We will follow the same steps to derive the forward price of specific collateral and the specific-collateral term repo rate. Let $F_\alpha(t,\tau)$ denote the forward price at time t of specific collateral α to be delivered at time τ, where $t \le \tau \le T_\alpha$. Again, by definition, the forward price makes the value of the forward contract zero:

$$E_t^{\mathcal{Q}}\left[\psi(t,\tau)\Big(P_\alpha(\tau) - F_\alpha(t,\tau)\Big)\right] = 0. \tag{2.8}$$

Solving (2.8) for the forward price yields

$$F_\alpha(t,\tau) = \frac{V_\alpha(t,\tau)}{P(t,\tau)}, \tag{2.9}$$

where

$$V_\alpha(t,\tau) := E_t^{\mathcal{Q}}\left[\psi(t,\tau)P_\alpha(\tau)\right] \tag{2.10}$$

is the value of an asset that pays $P_\alpha(\tau)$ at time τ.[7] We will refer to the asset that pays $P_\alpha(\tau)$ at time τ as deferred specific collateral. Note that because deferred specific collateral does not receive any repo earnings prior to time τ, $V_\alpha(t,\tau) \le P_\alpha(t)$. Also compare (2.10) and (2.2).

Again, the specific-collateral term repo rate on bond α, $r_\alpha(t,\tau)$, is implicitly defined by

$$P_\alpha(t)\,\exp(r_\alpha(t,\tau) \times (\tau - t)) = F_\alpha(t,\tau). \tag{2.11}$$

Solving (2.11) for $r_\alpha(t,\tau)$ and using (2.7) and (2.9) gives

$$r_\alpha(t,\tau) = r(t,\tau) + \log\left(\frac{V_\alpha(t,\tau)}{P_\alpha(t)}\right) \Big/ (\tau - t). \tag{2.12}$$

The difference between the general-collateral and special-collateral term rates defines the term repo spread:

$$s_\alpha(t,\tau) := r(t,\tau) - r_\alpha(t,\tau).$$

It is sometimes more convenient to refer to $S_\alpha(t,\tau) := s_\alpha(t,\tau) \times (\tau - t)$, the term repo spread per term rather than per year. Note that from (2.12) we see that

$$S_\alpha(t,\tau) = \log\left(\frac{P_\alpha(t)}{V_\alpha(t,\tau)}\right).$$

Note that $\pi_\alpha(t) = S_\alpha(t, T_\alpha)$, so that the price premium defined in (2.3) can be thought of as the repo spread to maturity.

Before we leave this section, we briefly note how forward positions are established in the repo market. For expositional simplicity, we assume no haircuts for credit risk. A long forward position is established by buying the security outright in the spot market and simultaneously entering a term repo agreement. The payment received

[7]See Section 3 for the relationship between correctly calculated forward prices and naively calculated pseudo forward prices.

on the repo agreement equals the purchase price, so there is no net payment to establish the position. At the end of the term, you receive the collateral and pay the previously agreed amount. The maturity of the term repo is the delivery date. Similarly, a short forward position is established by reversing in the collateral for a fixed term and simultaneously selling the collateral in the spot market. The proceeds from the sale match the value of the collateral borrowed, so again there is no net payment to establish the position. At the end of the term you are obligated to deliver the security for which you receive the previously agreed amount. This method of establishing forward positions is consistent with either (2.6) or (2.11).

3. Pseudo forward prices

In Section 2 we derived the forward price of specific collateral from first principles. We saw that the effects of specialness complicate the calculation of forward prices. Consider the case of calculating the forward price for delivery three months hence of a three-month Treasury bill. The naively-calculated forward price would be the ratio of the price of the current six-month bill to the price of the current three-month bill without regard for the potential repo earnings of either bill during the next three months.

Let us consider more formally this *pseudo* forward price. Let β refer to a zero-coupon bond with potential repo earnings and maturity $T_\beta = \tau$. The pseudo forward price of bond α, $\widetilde{F}_\alpha(t,\tau)$, is defined as

$$\widetilde{F}_\alpha(t,\tau) := \frac{P_\alpha(t)}{P_\beta(t)}. \tag{3.1}$$

Recall that the true forward price is given by $F_\alpha(t,\tau) = V_\alpha(t,\tau)/P(t,\tau)$. The potential for repo earnings implies both $P_\alpha(t) \geq V_\alpha(t,\tau)$ and $P_\beta(t) \geq P(t,\tau)$. Thus we cannot say in advance whether the pseudo forward price is higher or lower on average than the true forward price.

Noting that $P(t,\tau) = V_\beta(t,\tau)$, we can write

$$\begin{aligned}\widetilde{F}_\alpha(t,\tau) &= \exp\Big(S_\alpha(t,\tau) - S_\beta(t,\tau)\Big)\, F_\alpha(t,\tau) \\ &= \exp\Big((r_\alpha(t,\tau) - r_\beta(t,\tau)) \times (\tau - t)\Big) F_\alpha(t,\tau).\end{aligned} \tag{3.2}$$

Equation (3.2) shows that $\widetilde{F}_\alpha(t,\tau)$ involves the term special rate on security β, which matures at time τ, and differs from the true forward price unless the term special rates to $T_\beta = \tau$ on securities α and β are the same.[8]

The use of pseudo forward prices in place of true forward prices can lead to spurious rejections of empirical hypotheses. Consider the following two examples. For the first example, suppose the hypothesis is that the forward price is the conditional expectation of the future spot price under the physical measure, $\mathcal{P}$:

[8] In fact, if one were to define a pseudo term special rate for α in terms of the pseudo forward price,

$$\tilde{r}_\alpha(t,\tau) := \frac{1}{\tau - t}\,\log\left(\frac{\widetilde{F}_\alpha(t,\tau)}{P_\alpha(t)}\right) = r_\beta(t,\tau),$$

it would be the specific-collateral rate on the wrong security!

$F_\alpha(t,\tau) = E_t^{\mathcal{P}}[P_\alpha(\tau)]$. Combining this expression with (3.2) yields the relationship between pseudo forward price and the expected future spot price:

$$E_t^{\mathcal{P}}\left[P_\alpha(\tau)\right] = \exp\Big(S_\beta(t,\tau) - S_\alpha(t,\tau)\Big)\widetilde{F}_\alpha(t,\tau). \tag{3.3}$$

From (3.3), we see that there is a time varying bias in the relationship between $E_t^{\mathcal{P}}[P_\alpha(\tau)]$ and $\widetilde{F}_\alpha(t,\tau)$. Equation (3.3) suggests a reason why researchers might find time-varying term premiums in Treasury security prices even if they were not there.[9]

For the second example, we compare the pseudo forward price, $\tilde{F}_\alpha(t,\tau)$, to the specific-collateral futures price, $\mathcal{F}_\alpha(t,\tau)$. The specific-collateral futures price is given by

$$\mathcal{F}_\alpha(t,\tau) = E_t^{\mathcal{Q}}[P_\alpha(\tau)], \tag{3.4}$$

and the relationship between the true forward price and the futures price is given by

$$F_\alpha(t,\tau) = \mathcal{F}_\alpha(t,\tau) + \frac{\operatorname{cov}_t^{\mathcal{Q}}[\psi(t,\tau),\ P_\alpha(\tau)]}{P(t,\tau)}, \tag{3.5}$$

where $\operatorname{cov}_t^{\mathcal{Q}}[\,\cdot\,,\,\cdot\,]$ is the conditional covariance. Many studies of the Treasury bill market have found that the difference between forward and futures prices are larger than can be reasonably accounted for by the covariance term. Moreover, these studies have found that the futures price is a better predictor of the future spot price than the forward price. These studies,[10] however, have used the pseudo forward price rather than the true forward price. The relationship between the futures price and the pseudo-forward price is given by

$$\mathcal{F}_\alpha(t,\tau) = \exp\Big(S_\beta(t,\tau) - S_\alpha(t,\tau)\Big)\widetilde{F}_\alpha(t,\tau) - \frac{\operatorname{cov}_t^{\mathcal{Q}}[\psi(t,\tau),\ P_\alpha(\tau)]}{P(t,\tau)}. \tag{3.6}$$

Equation (3.6) shows that the difference between the pseudo forward price and the futures price could be greater than has been generally appreciated.[11]

4. Forward rates and the absence of arbitrage

In this section we now examine forward repo rates and forward repo spreads in a framework similar to that of Heath, Jarrow, and Morton (1992). In particular, we derive absence-of-arbitrage conditions in terms of forward repo rates and spreads.

[9]See Backus, Gregory, and Zin (1989) for an example of such tests and findings.
[10]There is a large literature examining the empirical relationship between forwards and futures in the Treasury bill market. See for example Fried (1994), Kamara (1988), and Krehbiel and Adkins (1994) and the papers cited therein. Some studies have recognized the importance of *financing costs*; see for example Gendreau (1985) and Kawaller and Koch (1984).
[11]We do not pursue these issues further in this paper as that would take us away from our central purpose of characterizing the term structure of repo spreads.

Forward repo rates. The relationship between general-collateral forward rates and general-collateral forward prices is

$$f(t,\tau) := \frac{\partial}{\partial \tau} \log(F(t,\tau,T)),$$

where $f(t,\tau)$ is the instantaneous forward rate at time t for term $\tau \geq t$. Thus, given $F(t,T,T) \equiv 1$, we can write

$$F(t,\tau,T) = \exp\left(-\int_{u=\tau}^{T} f(t,u)\,du\right). \tag{4.1}$$

Of course, $P(t,T) \equiv F(t,t,T)$. There is an expression analogous to (4.1) for specific collateral. Define the forward repo rate on bond α for term τ as:

$$f_\alpha(t,\tau) := \frac{\partial}{\partial \tau} \log\left(F_\alpha(t,\tau)\right),$$

which, using $F_\alpha(t,T_\alpha) \equiv 1$, implies that

$$F_\alpha(t,\tau) = \exp\left(-\int_{u=\tau}^{T_\alpha} f_\alpha(t,u)\,du\right). \tag{4.2}$$

Again, $P_\alpha(t) \equiv F_\alpha(t,t)$.

Absence of arbitrage. Following HJM, we assume the process for forward rates is given by

$$df(t,\tau) = \mu_f(t,\tau)\,dt + \sigma_f(t,\tau)^\top dW(t),$$

where $W(t)$ is a vector of orthonormal Brownian motions (and $^\top$ denotes scalar product). The process for the price of general collateral, $P(t,T)$, follows from that for forward rates by noting that $P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right)$; simply apply Itô's lemma to obtain

$$dP(t,T)/P(t,T) = \mu_P(t,T)\,dt + \sigma_P(t,T)^\top dW(t),$$

where

$$\sigma_P(t,T) = -\int_t^T \sigma_f(t,u)\,du, \tag{4.3a}$$

and

$$\mu_P(t,T) = r(t) + \frac{1}{2}\left\|\sigma_P(t,T)\right\|^2 - \int_t^T \mu_f(t,u)\,du. \tag{4.3b}$$

Under the physical measure, the standard no-arbitrage restriction for asset prices is

$$\mu_P(t,T) = r(t) + \sigma_P(t,T)^\top \lambda(t), \tag{4.4}$$

where $\lambda(t)$ is the market price of risk. To get the HJM restriction on the drift of forward rates (under the physical measure $\mathcal{P}$), differentiate (4.4) with respect to T and use (4.3):

$$\mu_f(t,T) = \sigma_f(t,T)^\top \left(\lambda(t) + \int_t^T \sigma_f(t,u)\,du\right). \tag{4.5}$$

Now we turn to the restrictions that the absence of arbitrage imposes on specific-collateral forward rates and forward repo spreads. We exploit the relationship between forward prices and forward rates for specific collateral given by (4.2). Let the process for special-collateral forward rates be given by

$$df_\alpha(t,\tau) = \mu_{f_\alpha}(t,\tau)\,dt + \sigma_{f_\alpha}(t,T)^\top dW(t).$$

Applying Itô's lemma to (4.2) and using the process for f_α, the process for the specific-collateral forward price is

$$dF_\alpha(t,\tau)/F_\alpha(t,\tau) = \mu_{F_\alpha}(t,\tau)\,dt + \sigma_{F_\alpha}(t,\tau)^\top dW(t),$$

where[12]

$$\sigma_{F_\alpha}(t,\tau) = -\int_\tau^{T_\alpha} \sigma_{f_\alpha}(t,u)\,du, \tag{4.6a}$$

and

$$\mu_{F_\alpha}(t,\tau) = \frac{1}{2}\,\|\sigma_{F_\alpha}(t,\tau)\|^2 - \int_\tau^{T_\alpha} \mu_{f_\alpha}(t,u)\,du. \tag{4.6b}$$

The forward price is not the value of an asset, and thus its relative drift is not given directly by the absence of arbitrage condition. It is however the ratio of two asset prices; therefore, apply Itô's lemma again, this time to $F_\alpha(t,\tau) = V_\alpha(t,\tau)/P(t,\tau)$, to obtain

$$\sigma_{F_\alpha}(t,\tau) = \sigma_{V_\alpha}(t,\tau) - \sigma_P(t,\tau), \tag{4.7a}$$

and

$$\mu_{F_\alpha}(t,\tau) = \mu_{V_\alpha}(t,\tau) - \mu_P(t,\tau) - \sigma_{F_\alpha}(t,\tau)^\top \sigma_P(t,\tau). \tag{4.7b}$$

Since $V_\alpha(t,\tau)$ is the value of an asset that pays no dividends, its expected rate of price appreciation equals its expected rate of return:

$$\mu_{V_\alpha}(t,\tau) = r(t) + \sigma_{V_\alpha}(t,\tau)^\top \lambda(t). \tag{4.8}$$

Using (4.8) and (4.7a), we can rewrite (4.7b) as

$$\mu_{F_\alpha}(t,\tau) = \sigma_{F_\alpha}(t,\tau)^\top \Big(\lambda(t) - \sigma_P(t,\tau)\Big). \tag{4.9}$$

We can translate (4.9) into a restriction on the drift of the forward rate process for specific collateral. Differentiating (4.9) with respect to τ and using (4.3) and (4.6) yields the following restriction:

$$\begin{aligned} \mu_{f_\alpha}(t,\tau) = \sigma_{f_\alpha}(t,\tau)^\top &\left(\lambda(t) + \int_t^\tau \sigma_f(t,u)\,du\right) \\ &- \Big(\sigma_f(t,\tau) - \sigma_{f_\alpha}(t,\tau)\Big)^\top \int_\tau^{T_\alpha} \sigma_{f_\alpha}(t,u)\,du \end{aligned} \tag{4.10}$$

[12]Notice that the indices of the integrals in (4.6) run from τ to T_α rather than from t to τ as they do in (4.3).

The restriction of the drift of the process for specific-collateral forward repo rates given by (4.10) is somewhat complicated. Some simplification can be achieved by focusing on the restriction for forward repo spreads.

Forward repo spreads. With general-collateral and specific-collateral forward curves both available, we define the forward repo spread curve by

$$\delta_\alpha(t,\tau) := f(t,\tau) - f_\alpha(t,\tau).$$

The process for forward repo spreads follows from those for general and special collateral

$$\begin{aligned} d\delta_\alpha(t,\tau) &= \mu_{\delta_\alpha}(t,\tau)\,dt + \sigma_{\delta_\alpha}(t,\tau)^\top dW(t) \\ &= (\mu_f(t,\tau) - \mu_{f_\alpha}(t,\tau))\,dt + (\sigma_f(t,\tau) - \sigma_{f_\alpha}(t,\tau))^\top dW(t). \end{aligned}$$

Since the forward repo spread must always lie within the support of the spot repo spread, and since $s_\alpha(t) \geq 0$, we have the following restriction:

$$\delta_\alpha(t,\tau) \geq 0 \tag{4.11}$$

for all $\tau \geq t$. We can use (4.5) and (4.10) to obtain the drift for the forward spread process:

$$\mu_{\delta_\alpha}(t,\tau) = \sigma_{\delta_\alpha}(t,\tau)^\top \left(\lambda(t) + \int_t^{T_\alpha} \sigma_f(t,u)\,du - \int_\tau^{T_\alpha} \sigma_{\delta_\alpha}(t,u)\,du\right). \tag{4.12}$$

Expression (4.12), in combination with (4.11), is our characterization of the restriction on the forward repo spread process.[13]

It may be useful to rewrite (4.12) as follows:

$$\mu_{\delta_\alpha}(t,\tau) = \sigma_{\delta_\alpha}(t,\tau)^\top \left(\lambda(t) - \int_\tau^{T_\alpha} \sigma_{\delta_\alpha}(t,u)\,du\right) - \sigma_{\delta_\alpha}(t,\tau)^\top \sigma_P(t,T_\alpha). \tag{4.13}$$

From (4.13) we see that the restriction on the drift on the spread process is similar in form to the restriction on the drift for general-collateral forward rates given by (4.5)—except for the term involving the instantaneous covariance between the time-τ forward spread and the return on general collateral that matures at time T_α.

5. The unbiased expectations hypotheses

In this section, we derive some implications of the unbiased expectations hypotheses (U–EH) for repo spreads, and we characterize the U–EH in terms of the drift of the forward repo spread process.

[13] The example in Duffie (1996b) fits into this framework. In his example, there is but one Brownian so that repo-spread risk is either identically zero or perfectly correlated with general-collateral risk.

Characterizing the U–EH. We define the forward repo spread term premium as follows:

$$\gamma_\alpha(t,\tau) := \delta_\alpha(t,\tau) - E_t^{\mathcal{P}}[s_\alpha(\tau)]. \tag{5.1}$$

The U–EH asserts that $\gamma_\alpha(t,\tau) \equiv 0$. Next, note that the overnight repo spread is related to the forward repo spread curve by

$$s_\alpha(\tau) \equiv \delta_\alpha(\tau,\tau) = \delta_\alpha(t,\tau) + \int_{v=t}^{\tau} d\delta_\alpha(v,\tau). \tag{5.2}$$

Taking conditional expectations of both sides under the physical measure produces

$$E_t^{\mathcal{P}}[s_\alpha(\tau)] = \delta_\alpha(t,\tau) + E_t^{\mathcal{P}}\left[\int_{v=t}^{\tau} d\delta_\alpha(v,u)\right] = \delta_\alpha(t,\tau) + \int_{v=t}^{\tau} E_t^{\mathcal{P}}[\mu_{\delta_\alpha}(v,\tau)]\, dv.$$

Thus we can write the forward spread term premium in terms of the expected drift of the spread process:

$$\gamma_\alpha(t,\tau) = \int_{v=t}^{\tau} E_t^{\mathcal{P}}[-\mu_{\delta_\alpha}(v,\tau)]\, dv. \tag{5.3}$$

From (5.3) we see that the U–EH for repo spreads requires that

$$\mu_{\delta_\alpha}(t,\tau) \equiv 0. \tag{5.4}$$

In other words, forward repo spreads must be martingales under the U–EH.

Fisher and Gilles (1996) show how to construct examples of the U–EH for general collateral, where forward general-collateral rates are martingales. They define an auxiliary function

$$\xi(t,\tau) := \lambda(t) + \int_{s=t}^{\tau} \sigma_f(t,s)\, ds.$$

Note that $\xi'(t,\tau) := (\partial/\partial\tau)\,\xi(t,\tau) = \sigma_f(t,\tau)$. With this auxiliary function, the HJM forward rate drift restriction (4.5) under the U–EH ($\mu_f(t,\tau) \equiv 0$) can be written as

$$\xi'(t,\tau)^\top \xi(t,\tau) \equiv 0.$$

Thus as long as $\|\xi(t,\tau)\|$ is constant, the U–EH will be satisfied. To construct an example, simply pick a function $\xi(t,\tau)$ that is a rotation of $\xi(t,t) = \lambda(t)$ and define $\sigma_f(t,\tau) = \xi'(t,\tau)$.

We can apply the same technique to forward repo spreads. In particular, define

$$\xi_\alpha(t,\tau) := \lambda(t) + \int_{t}^{T_\alpha} \sigma_f(t,u)\, du - \int_{\tau}^{T_\alpha} \sigma_{\delta_\alpha}(t,u)\, du. \tag{5.5}$$

Note that $\xi'_\alpha(t,\tau) = \sigma_{\delta_\alpha}(t,\tau)$. Equation (5.4) can be reduced to

$$\xi'_\alpha(t,\tau)^\top \xi_\alpha(t,\tau) \equiv 0.$$

Thus, as long as $\|\xi_\alpha(t,\tau)\|$ is constant, the U–EH for repo spreads will be satisfied. Notice that there must be at least two sources of risk for this to be possible. Again, to construct an example, pick a function $\xi_\alpha(t,\tau)$ that is a rotation of $\xi_\alpha(t,T_\alpha) =$

$\lambda(t) + \int_t^{T_\alpha} \sigma_f(t,u)\,du$ and define $\sigma_{\delta_\alpha}(t,\tau) = \xi'_\alpha(t,\tau)$.[14] There remains a problem however: We have not been able to discover a candidate $\xi_\alpha(t,\tau)$ in which the non-negativity of $s_\alpha(t)$ can be guaranteed.[15] Therefore, it remains an open question as to whether the U–EH for repo spreads can hold absent arbitrage opportunities.

Implications of the U–EH. Notwithstanding the foregoing open question, we will derive here some testable implications of the U–EH. Note that the term repo spread can be expressed in terms of the forward repo spread curve:

$$S_\alpha(t,\tau) = \int_{u=t}^{\tau} \delta_\alpha(t,u)\,du. \tag{5.6}$$

Using (5.2) and (5.6) we can show that

$$\int_{u=t}^{\tau} E_t^{\mathcal{P}}\left[s_\alpha(u)\right]\,du = S_\alpha(t,\tau) + \int_{u=t}^{\tau}\int_{v=t}^{u} E_t^{\mathcal{P}}\left[\mu_\delta(v,u)\right]\,dv\,du \tag{5.7a}$$

and

$$E_t^{\mathcal{P}}\left[S_\alpha(t',\tau)\right] = S_\alpha(t,\tau) - S_\alpha(t,t') + \int_{u=t'}^{\tau}\int_{v=t}^{t'} E_t^{\mathcal{P}}\left[\mu_\delta(v,u)\right]\,dv\,du, \tag{5.7b}$$

for $t \le t' \le \tau$.[16] Under the U–EH, we can write (5.7) as

$$\int_{u=t}^{\tau} E_t^{\mathcal{P}}\left[s_\alpha(u)\right]\,du = S_\alpha(t,\tau) \tag{5.8a}$$

and

$$E_t^{\mathcal{P}}\left[S_\alpha(t',\tau)\right] = S_\alpha(t,\tau) - S_\alpha(t,t'). \tag{5.8b}$$

Note that we can rewrite (5.8b) as

$$E_t^{\mathcal{P}}\left[s_\alpha(t',\tau)\right] - s_\alpha(t,\tau) = \left(\frac{t'-\tau}{\tau-t'}\right)\Big(s_\alpha(t,\tau) - s_\alpha(t,t')\Big), \tag{5.8b$'$}$$

[14] Note that the U–EH can hold for repo spreads even if it does not hold for general collateral.

[15] See (Fisher and Gilles 1996) for a discussion of the analogous difficulty of finding a model of the U–EH for general collateral that guarantees the non-negativity of $r(t)$.

[16] Equation (5.7a) follows from integrating (5.2) and using (5.6), while equation (5.7b) follows from (5.6)

$$E_t^{\mathcal{P}}\left[S_\alpha(t',\tau)\right] = \int_{u=t'}^{\tau} E_t^{\mathcal{P}}\left[\delta_\alpha(t',u)\right]du,$$

from (5.2)

$$E_t^{\mathcal{P}}\left[\delta_\alpha(t',u)\right] = \delta_\alpha(t,u) + \int_{v=t}^{t'} E_t^{\mathcal{P}}\left[\mu_{\delta_\alpha}(v,u)\right]dv,$$

and from (5.6) again

$$\int_{u=t'}^{\tau} \delta_\alpha(t,u)\,du = S_\alpha(t,\tau) - S_\alpha(t,t').$$

which is analogous to one of the relationships for general collateral that Campbell and Shiller (1991) examine. Letting $\tau \to \tau_\alpha$, (5.8) becomes

$$\int_{u=t}^{\tau_\alpha} E_t^{\mathcal{P}}\left[s_\alpha(u)\right] du = \pi_\alpha(t) \tag{5.9a}$$

and

$$E_t^{\mathcal{P}}\left[\pi_\alpha(t')\right] = \pi_\alpha(t) - S_\alpha(t,t'). \tag{5.9b}$$

In Section 7 we subject (5.8a), (5.8b′), and (5.9b) to empirical examination.

6. The auction cycle

The data. This section describes the data set of repo rates and Treasury prices upon which the empirical tests are based.[17]

The repo data are daily from 1987/03/31 through 1991/12/11. Repo rates for the following collateral are included: general collateral; on-the-run two-, three-, seven-, ten-, and thirty-year coupon securities; and old seven-, ten-, and thirty-year coupon securities. For each type of collateral the following maturities are included: overnight; one, two, and three weeks; and one, two, and three months. The repo rates are afternoon "readings." The rates were entered by the traders after things slow down. They do not represent actually trades. (Trading in the afternoon is thin and bid-ask spreads are wide.) Rather, the rates represent where the traders believed the market to be.

The Treasury quotes come from the Federal Reserve Bank of New York. They are the bid side of the "closing quotes" taken by a telephone survey at about 3:30 p.m. The price premiums are calculated relative to the value implied by daily estimated term structure using the techniques described in Fisher, Nychka, and Zervos (1995). The data base of Treasury prices begins on 1987/12/01.

The cycle. Each bond of a given original term to maturity is on a given auction cycle. The two-year note is on a one-month cycle, while the three-, seven-, ten-, and thirty-year securities are on three-month cycles. There are three important periodic dates: the announcement date, the auction date, and the settlement (or issuance) date. On the announcement date, the Treasury announces the particulars of the upcoming auction—in particular, the amount to be auctioned—and when-issued trading begins. On the auction date, the auction is held. On the settlement date the security is issued. There is typically about one week from the announcement to the auction and one week from the auction to the issuance. We have chosen to date the cycle relative to the announcement date *following* the issuance of the security. See Figure 1.

Figure 2 shows the average overnight repo spread by type of collateral by auction-cycle day. We see that the crisis for the average of overnight repo spreads occurs at time zero—when trading begins in the next security to be auctioned. Figure 3 shows the 25-th, 50-th, and 75-th quantiles for the overnight repo spreads by day by collateral. Term repo spreads are shown in Figure 4: one-, two-, and three-month

[17] The repo data are proprietary and may not be published in tabular form or made available. The source has requested anonymity.

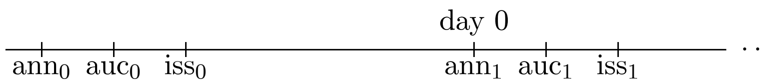

FIGURE 1. The auction cycle time line

term spreads are shown as solid, long-dashed, and short-dashed. For the seven-, ten-, and thirty-year securities, one can see the steeply downward sloping average spread curves at day -75, which become steeply upward sloping by day -25.

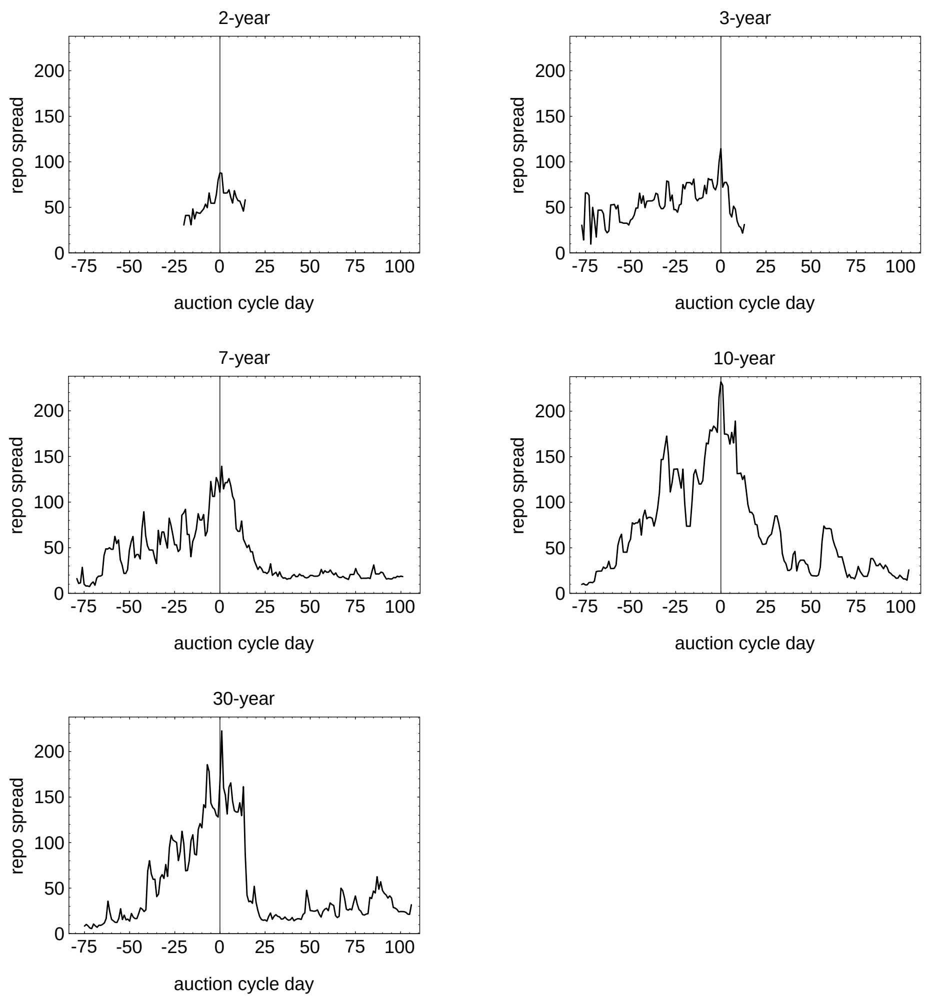

FIGURE 2. Average overnight repo spreads in basis points

Figure 5 shows the average price premium by type of collateral by auction-cycle day. Figure 6 shows the relationship between the average premium and the sum of the subsequent average overnight repo spreads. The overnight rates have been

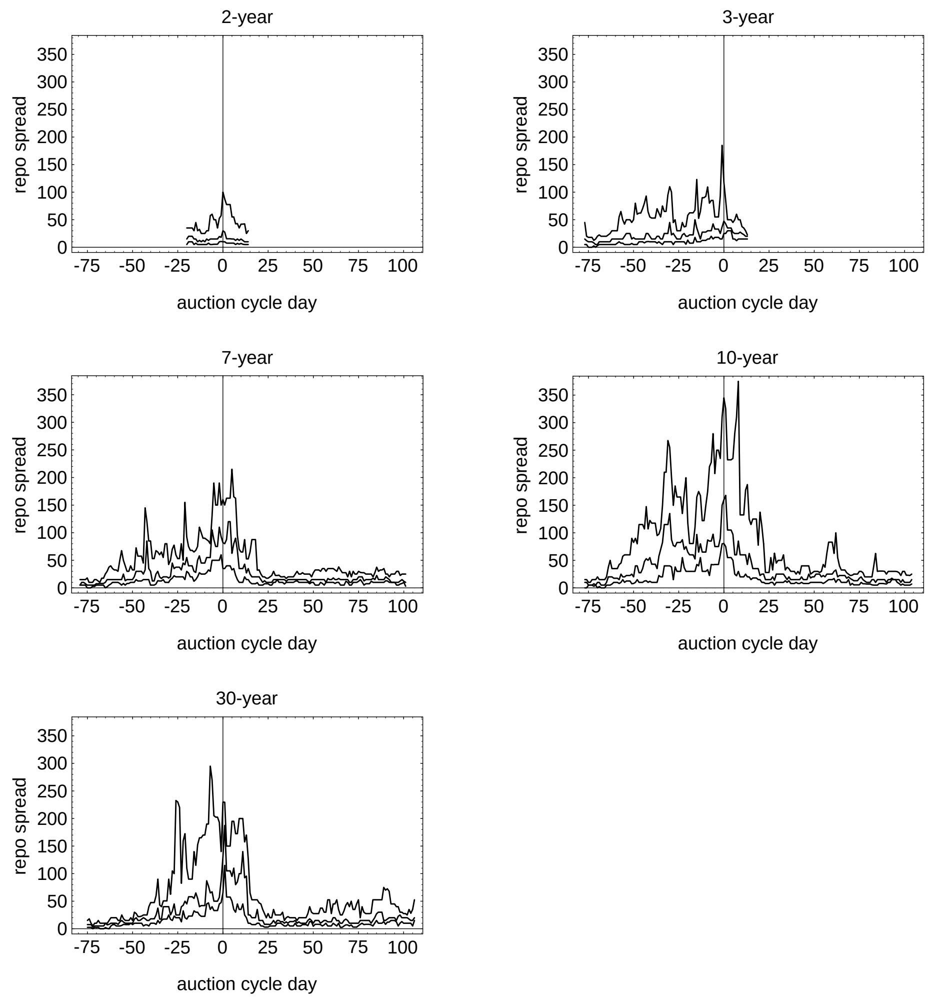

FIGURE 3. Quartiles for overnight repo spreads in basis points

raised by the mean of the price premiums. Figure 7 shows the same relationship as scatter-plots. The lines are not fit: they have slope one and go through the means of the two series.

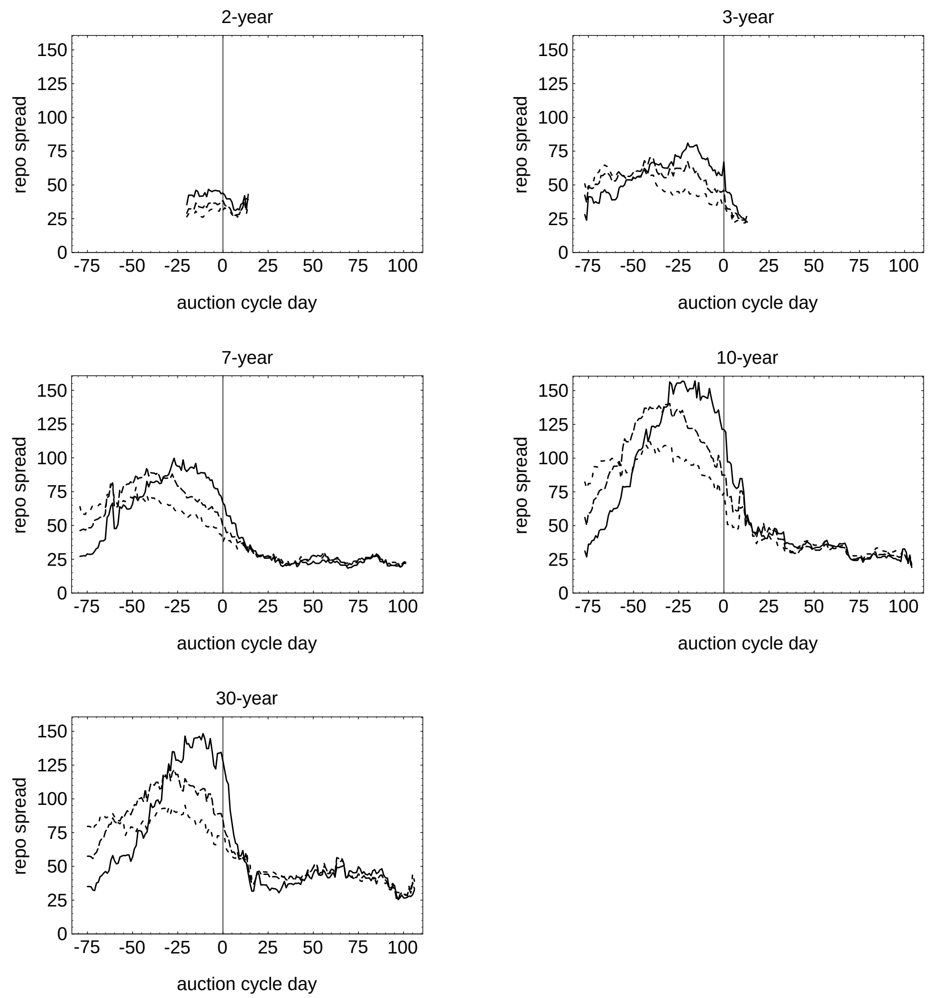

Figure 4. Average term repo spreads in basis points: 1-, 2-, and 3-months

7. Empirical results

In this section we develop tests of the U–EH and report results.[18] For convenience, we repeat the three relationships we intend to test:

$$\int_{u=t}^{\tau} E_t^{\mathcal{P}}\left[s_\alpha(u)\right]\, du = S_\alpha(t,\tau), \tag{7.1a}$$

$$E_t^{\mathcal{P}}\left[s_\alpha(t',\tau)\right] - s_\alpha(t,\tau) = \left(\frac{t'-\tau}{\tau-t'}\right)\Big(s_\alpha(t,\tau) - s_\alpha(t,t')\Big), \tag{7.1b}$$

[18]This section is incomplete.

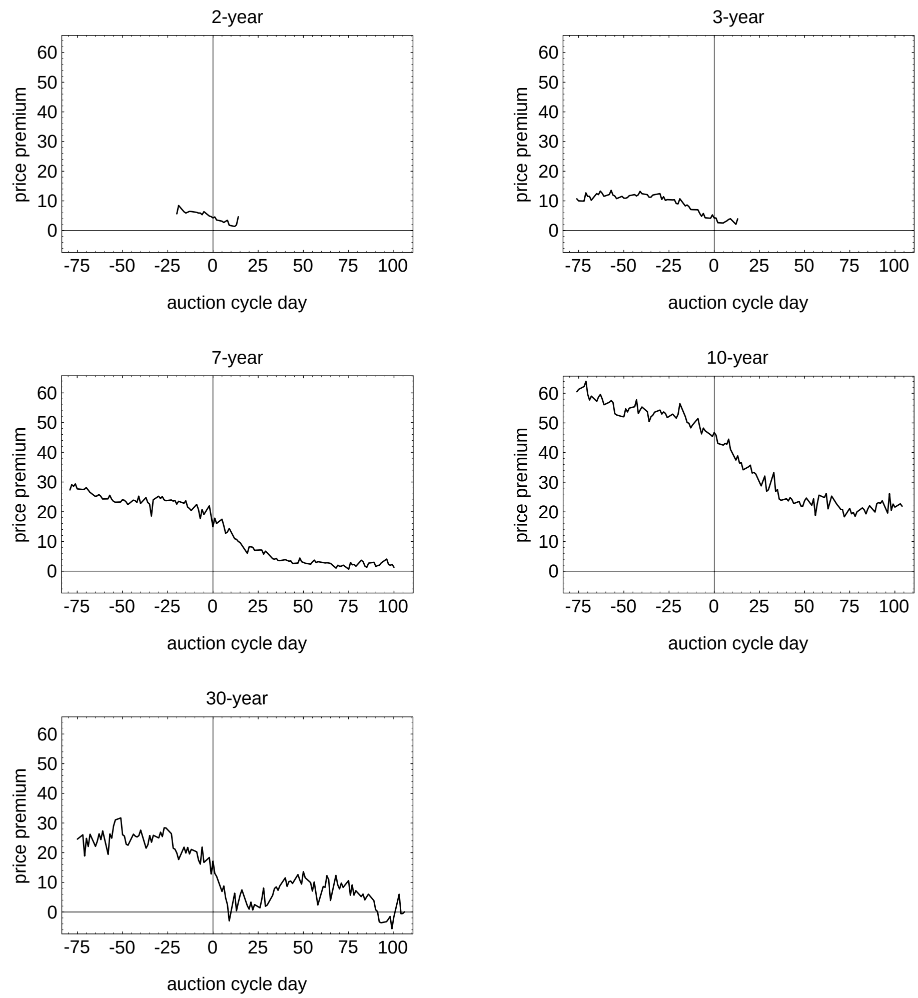

FIGURE 5. Average price premia in basis points

and

$$\pi_\alpha(t) - E_t^{\mathcal{P}}\left[\pi_\alpha(\tau)\right] = S_\alpha(t, \tau). \tag{7.1c}$$

These three relationships say that (i) the term spread forecasts the sum of the future overnight spreads, (ii) the slope of the spread curve forecasts the change in the spread, and (iii) the term spread forecasts the change in the price premium.

To give these relationships empirical content, we will assume that the realizations equal the forecast plus a rational-expectations forecast error term:

$$s_\alpha(u) = E_t^{\mathcal{P}}\left[s_\alpha(u)\right] + \varepsilon_u^1, \tag{7.2a}$$

$$s_\alpha(t', \tau) = E_t^{\mathcal{P}}\left[s_\alpha(t', \tau)\right] + \varepsilon_{t'}^2, \tag{7.2b}$$

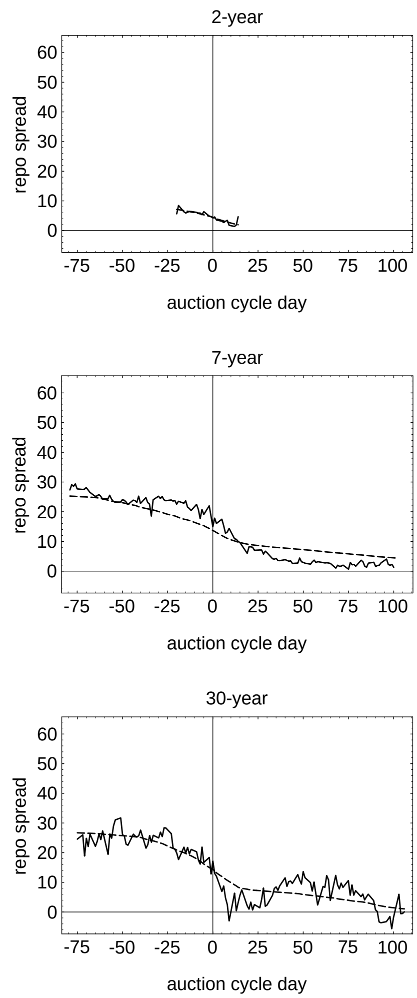

FIGURE 6. Accumulated overnight spreads and price premia

and

$$\pi_\alpha(\tau) = E_t^{\mathcal{P}}\left[\pi_\alpha(\tau)\right] + \varepsilon_\tau^3. \tag{7.2c}$$

APPENDIX A. COUPON BONDS

The analysis in this appendix extends that of Section 2 to apply to coupon bonds; although the extension is not exactly consistent with term repo contracts in U.S. markets, it provides both a starting point and a useful approximation. The only difficulty occurs when a coupon is paid during the life of the repo contract, because then expression (2.11) no longer holds. We seek a generalization that holds for coupon bonds.

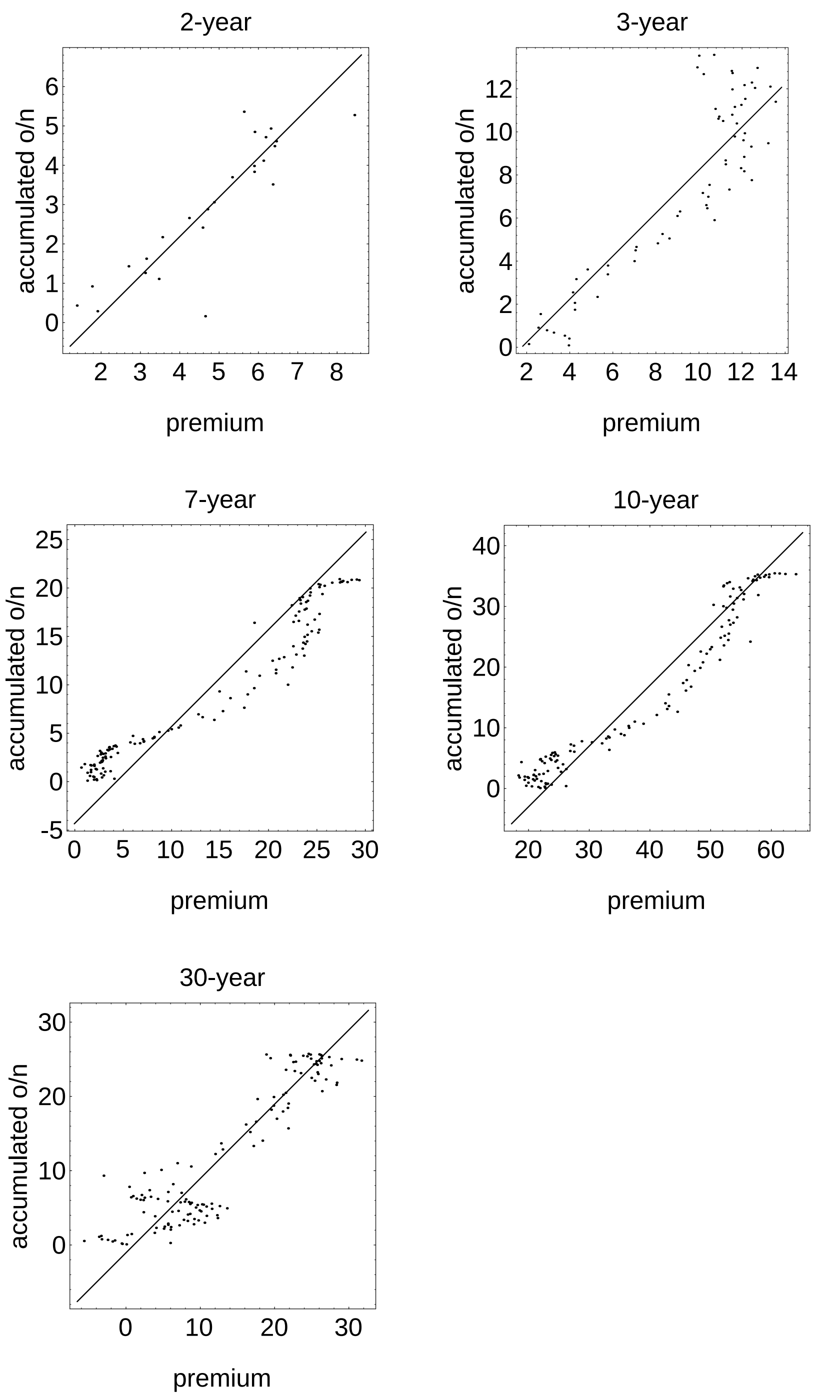

Figure 7. Accumulated overnight spreads versus price premia

Let $P_\alpha(t)$ be the price of a coupon bond at time t that matures at time T_α and pays (lumpy) coupon C_i at times t_i, $t_1 < t_2 < \ldots < t_n = T_\alpha$. Let t_0 ($t_0 < t_1$) be the time at which the repo contract is entered into. Define a new price process $\widehat{P}_\alpha(t)$ as follows

$$\widehat{P}_\alpha(t) := \theta(t) P_\alpha(t), \qquad \text{for all } t \geq t_0, \tag{A.1}$$

where $\theta(0) = 1$, $\theta(t)$ is constant on any half-open interval $[t_i, t_{i+1})$, and

$$\theta(t_{i+1}) := \theta(t_i) \frac{P_\alpha(t_i) + C_i}{P_\alpha(t_i)}, \qquad \text{for } i = 1, 2, \ldots, n. \tag{A.2}$$

Basically, (A.1) and (A.2) say that $\widehat{P}_\alpha(t)$ is the value of a self-financing portfolio at time t formed by buying one bond at time t_0 and reinvesting all of the lumpy coupon payments in the bond itself. Now, let $\widehat{F}_\alpha(t,\tau)$ be the forward price for delivery at time τ of this portfolio. Let the term special repo rate $r_\alpha(t,\tau)$ be implicitly defined by equation (2.11) with price $P_\alpha(t)$ replaced by $\widehat{P}_\alpha(t)$ and forward price $F_\alpha(t,\tau)$ replaced by $\widehat{F}_\alpha(t,\tau)$.

With these changes, the analysis in the text can proceed with no further qualification to coupon bonds. The resulting description of the equilibrium in the U.S. repo market is only an approximation because the analysis tacitly assumes that the original owner of the security borrows $P_\alpha(t_0)$ (an amount of money which includes the present value of all future coupons) at the special rate $r_\alpha(t_0,\tau)$, earning the term repo spread on the full amount borrowed for the full length of the term. In U.S. repo markets, though, the value of the loan is knocked down each time a coupon is paid; in effect, the owner cannot earn the repo spread on the coupons paid during the life of the term contract. Correctly calculated, the term repo rate $r_\alpha(t_0,\tau)$ is the solution for x in the following equation, which generalizes (2.11):

$$P_\alpha(t_0) \exp\left(x(\tau - t_0)\right) - \sum_{i=1}^{n} C_i \exp\left(x(\tau - t_i)\right) = F_\alpha(t_0, \tau).$$

Thus, the approximation error involves only the repo spread on the coupon paid during the life of the contract, and is tolerably small for the contracts in our data, which do not extend beyond six months, during which at most one coupon is paid.

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